

Hurwitz correspondences on compactifications of $\mathcal{M}_{0,N}$.

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Abstract

Hurwitz correspondences are certain multivalued self-maps of the moduli space $\mathcal{M}_{0,N}$. They arise in the study of Thurston's topological characterization of rational functions. We consider the dynamics of Hurwitz correspondences and ask: On which compactifications of $\mathcal{M}_{0,N}$ should they be studied? We compare a Hurwitz correspondence \mathcal{H} across various modular compactifications of $\mathcal{M}_{0,N}$, and find a weighted stable curves compactification X_N^\dagger that is optimal for its dynamics. We use X_N^\dagger to show that the k th dynamical degree of \mathcal{H} is the absolute value of the dominant eigenvalue of the pushforward induced by \mathcal{H} on a natural quotient of $H_{2k}(\overline{\mathcal{M}}_{0,N})$.

Keywords: Dynamical degrees, Hurwitz spaces, $\overline{\mathcal{M}}_{0,N}$, rational correspondences.

1. Introduction

We recall the moduli space $\mathcal{M}_{0,\mathbf{P}}$, a quasiprojective variety parametrizing smooth genus zero curves with marked points labeled by the elements of a finite set \mathbf{P} . Let \mathcal{H} be a *Hurwitz space* parametrizing maps, with prescribed branching, from one \mathbf{P} -marked genus zero curve to another. \mathcal{H} admits two maps to $\mathcal{M}_{0,\mathbf{P}}$ — a ‘target curve’ map π_1 , and a ‘source curve’ map π_2 . If all the branch values of the target curve are marked, π_1 is a covering map, so $\pi_2 \circ \pi_1^{-1}$ is a multivalued map — a *Hurwitz correspondence* — from $\mathcal{M}_{0,\mathbf{P}}$ to itself. We study the dynamics of Hurwitz correspondences via invariants called *dynamical degrees*.

Dynamical degrees are numerical invariants associated to a self-map of a smooth projective variety ([10, 28]). They measure dynamical complexity of iteration. For $g : X \rightarrow X$ a surjective regular map, the k th dynamical degree of g is the absolute value of the dominant eigenvalue of the pullback $g^* : H^{k,k}(X) \rightarrow H^{k,k}(X)$. For g a rational self-map or a rational multivalued self-map, we may not have $(g^n)^* = (g^*)^n$. In these cases the k th dynamical degree is defined to be

$$\lim_{n \rightarrow \infty} \|(g^n)^* : H^{k,k}(X) \rightarrow H^{k,k}(X)\|^{1/n}.$$

Suppose we do have $(g^n)^* = (g^*)^n$ on $H^{k,k}(X)$ — g may not be regular, but it behaves like a regular map in terms of its action on $H^{k,k}(X)$. Then g is called k -stable, and the k th dynamical degree of g is the absolute value of the dominant eigenvalue of $g^* : H^{k,k}(X) \rightarrow H^{k,k}(X)$. Computing or characterizing dynamical degrees of a map g which is *not* k -stable involves dealing with the pullbacks along infinitely many iterates g^n , and is notoriously difficult ([26, 1]). The dynamical degrees of a map provide information about a more fundamental invariant — topological entropy. The topological entropy of a regular map is the logarithm of the largest of its dynamical degrees ([14]); the topological entropy of a rational map or rational multivalued map is bounded from above by the logarithm of the largest of its dynamical degrees ([6, 7]).

$\mathcal{M}_{0,\mathbf{P}}$ is not compact — to make sense of the dynamical degrees of a Hurwitz correspondence \mathcal{H} we must consider it as a *rational* multivalued self-map of some projective compactification $X_{\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$. We are free to choose this compactification $X_{\mathbf{P}}$ for convenience — the dynamical degrees of \mathcal{H} are birational invariants

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([6, 7, 31, 25]) that purely reflect its dynamics on the interior, $\mathcal{M}_{0,\mathbf{P}}$. Koch and Roeder proved in [20] that some special examples are k -stable on the stable curves compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$, and computed their dynamical degrees. In this work, we exploit the moduli space interpretation of Hurwitz correspondences to provide an account of their homological actions on various compactifications of $\mathcal{M}_{0,\mathbf{P}}$. Together with a companion paper [25], this comprises the first systematic study of the dynamical degrees of any natural family of rational multivalued maps.

We first prove, jointly with Sarah Koch and David Speyer:

Proposition 8.1 (Koch, Ramadas, Speyer). *Let \mathcal{H} be a Hurwitz correspondence on $\mathcal{M}_{0,\mathbf{P}}$. Then for all k , \mathcal{H} is k -stable as a rational multivalued self-map of $\overline{\mathcal{M}}_{0,\mathbf{P}}$.*

Since $H^{k,k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) = H^{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$, and since pushforward and pullback maps are dual to each other (Section 4.1), we obtain:

Corollary 8.2 (Koch, Ramadas, Speyer). *The k th dynamical degree of \mathcal{H} is the absolute value of the dominant eigenvalue of the induced pushforward*

$$[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}).$$

In fact, $[\mathcal{H}]_*$ preserves the cone of effective classes in $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$, so it follows from the theory of cone-preserving operators that $[\mathcal{H}]_*$ has a nonnegative dominant eigenvalue, with a *pseudoeffective* eigenvector (Remark 8.3).

We use Harris and Mumford’s compactification of \mathcal{H} by a moduli space of *admissible covers* ([16]) to show that the multivalued map $\pi_2 \circ \pi_1^{-1}$, considered as a map from $\mathcal{M}_{0,\mathbf{P}}$ to $\text{Sym}^d(\mathcal{M}_{0,\mathbf{P}})$, extends to a *regular* map from $\overline{\mathcal{M}}_{0,\mathbf{P}}$ to $\text{Sym}^d(\overline{\mathcal{M}}_{0,\mathbf{P}})$. Thus, Hurwitz correspondences extend to $\overline{\mathcal{M}}_{0,\mathbf{P}}$ “without indeterminacy.” Proposition 8.1 follows as a consequence.

We next introduce a filtration $\{\Lambda_{\mathbf{P}}^{\leq \lambda}\}_\lambda$ of $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ indexed by the partially ordered set $\{\text{partitions } \lambda \text{ of } k\}$ (Section 9). We show:

Theorem 9.6. *Let \mathcal{H} be any Hurwitz correspondence on $\mathcal{M}_{0,\mathbf{P}}$. Then $[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ sends each subspace $\Lambda_{\mathbf{P}}^{\leq \lambda}$ to itself.*

Thus the operator $[\mathcal{H}]_*$ can be written, in multiple different ways, as a block-lower-triangular matrix. The blocks give the induced action of $[\mathcal{H}]_*$ on successive quotients of subspaces of the filtration $\{\Lambda_{\mathbf{P}}^{\leq \lambda}\}$.

Which of these blocks contains the dominant eigenvalue — the dynamical degree? Set

$$\Lambda_{\mathbf{P}}^{<(k)} := \sum_{\Lambda_{\mathbf{P}}^{\leq \lambda} \subsetneq H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})} \Lambda_{\mathbf{P}}^{\leq \lambda}$$

and

$$\Omega_{\mathbf{P}}^k := H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) / \Lambda_{\mathbf{P}}^{<(k)}.$$

Then $\Omega_{\mathbf{P}}^k$ is the smallest possible nonzero quotient of $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ obtainable by subspaces in the filtration $\{\Lambda_{\mathbf{P}}^{\leq \lambda}\}$. The induced action $[\mathcal{H}]_* : \Omega_{\mathbf{P}}^k \rightarrow \Omega_{\mathbf{P}}^k$ gives the topmost block of the block-lower-triangular matrix $[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$. We show:

Theorem 10.6. *The k th dynamical degree of \mathcal{H} is the absolute value of the dominant eigenvalue of $[\mathcal{H}]_* : \Omega_{\mathbf{P}}^k \rightarrow \Omega_{\mathbf{P}}^k$.*

This considerably increases the efficiency of computing dynamical degrees; for example, when $k = \dim \mathcal{M}_{0,\mathbf{P}} - 1$, the dimension of $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ is $\frac{2^{|\mathbf{P}|} - |\mathbf{P}|^2 + |\mathbf{P}| - 2}{2}$, while the dimension of $\Omega_{\mathbf{P}}^k$ is only $|\mathbf{P}|$.

We prove Theorem 10.6 by finding an alternate compactification $X_{\mathbf{P}}^\dagger$ of $\mathcal{M}_{0,\mathbf{P}}$ (a Hassett moduli space of *weighted* stable curves) with a birational morphism $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \rightarrow X_{\mathbf{P}}^\dagger$ that contracts the cycles in $\Lambda_{\mathbf{P}}^{<(k)}$.

Interestingly, for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$, the *only* k -cycles contracted by ρ are those in $\Lambda_{\mathbf{P}}^{<(k)}$, and although Hurwitz correspondences do not extend to $X_{\mathbf{P}}^{\dagger}$ without indeterminacy, they are k -stable on $X_{\mathbf{P}}^{\dagger}$.

$\overline{\mathcal{M}}_{0,\mathbf{P}}$ has “enough boundary” to resolve any potential indeterminacy of \mathcal{H} . However, Theorems 9.6 and 10.6 together imply that the boundary $\overline{\mathcal{M}}_{0,\mathbf{P}} \setminus \mathcal{M}_{0,\mathbf{P}}$, and the homology groups of $\overline{\mathcal{M}}_{0,\mathbf{P}}$, are too large. The action of $[\mathcal{H}]_*$ on $\Omega_{\mathbf{P}}^k$ is “intrinsic” to the dynamics of \mathcal{H} on $\mathcal{M}_{0,\mathbf{P}}$ — this action determines the dynamical degree, and for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$ is isomorphic to the k -stable homological action of \mathcal{H} on the birational model $X_{\mathbf{P}}^{\dagger}$. In contrast, the action on the subspace $\Lambda_{\mathbf{P}}^{<(k)}$ is an artifact of the compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$ — this action is “contained in the boundary” (Corollary 10.8). We note that Theorems 9.6 and 10.6 do not help give an upper bound for the entropy of \mathcal{H} . In ([25]), we show that for any Hurwitz correspondence \mathcal{H} , the sequence $k \mapsto (k\text{th dynamical degree of } \mathcal{H})$ is nonincreasing. Thus the largest dynamical degree — the one providing an upper bound for entropy — is the 0th. This is the topological degree of the ‘target curve’ map $\pi_1 : \mathcal{H} \rightarrow \mathcal{M}_{0,\mathbf{P}}$; a *Hurwitz number*.

Our study is partly motivated by a connection to Teichmüller theory. Hurwitz correspondences arise in the context of a criterion given by W. Thurston ([8]) for the existence of an algebraic self-map of \mathbb{P}^1 of a given topological type. Let $\phi : S^2 \rightarrow S^2$ be an orientation-preserving branched covering from a topological 2-sphere to itself. When is ϕ conjugate, up to homotopy, to a rational function $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$? Suppose ϕ has finite *post-critical set*

$$\mathbf{P} := \{\phi^n(x) | n > 0, x \text{ a critical point of } \phi\}.$$

Denote by $\mathcal{T}(S^2, \mathbf{P})$ the *Teichmüller space* parametrizing complex structures on the marked sphere (S^2, \mathbf{P}) . $\mathcal{T}(S^2, \mathbf{P})$ is a nonalgebraic complex manifold, and the universal cover of $\mathcal{M}_{0,\mathbf{P}}$. Given one complex structure on (S^2, \mathbf{P}) , one may pull it back along the branched covering ϕ to obtain another. This defines a holomorphic self-map of $\mathcal{T}(S^2, \mathbf{P})$ called the Thurston pullback map (Section 5.1). A fixed point of the Thurston pullback map is a complex structure under which ϕ is identified with a rational function $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Although the Thurston pullback map does not descend to a self-map of $\mathcal{M}_{0,\mathbf{P}}$, it does descend to a multivalued self-map — in fact, a Hurwitz correspondence (Koch, [19]). The dynamics of Hurwitz correspondences thus algebraically record the dynamics of the nonalgebraic Thurston pullback map.

Outline. Sections 4 through 7 give background on, in order, rational correspondences, Hurwitz correspondences, compactifications of $\mathcal{M}_{0,N}$, and the admissible covers compactifications of Hurwitz spaces. Our results are in Sections 8, 9, and 10. In Section 8, we show that Hurwitz correspondences are k -stable on $\overline{\mathcal{M}}_{0,N}$. In Section 9, we show they preserve a natural filtration of $H_{2k}(\overline{\mathcal{M}}_{0,N})$. In Section 10, we use alternate compactifications of $\mathcal{M}_{0,N}$ to investigate where in this filtration the dynamical degree lies.

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3. Conventions

All varieties and schemes are over \mathbb{C} . For X a smooth projective variety, we denote by $H_c(X)$ and $H^c(X)$ its c th singular homology and cohomology groups respectively in the analytic topology with coefficients in \mathbb{R} . There are “Poincaré duality” isomorphisms between $H^c(X)$ and $H_{\dim_{\mathbb{R}} X - c}(X)$. These allow us to define pushforward maps on cohomology groups and pullback maps on homology groups, and cup product of homology classes. For X any variety, we denote by $A_k(X)$ the Chow group of k -cycles on X up to rational equivalence.

All curves in this paper are projective reduced curves, connected but not necessarily irreducible, of arithmetic genus zero.

A partition λ of a positive integer k is a multiset of positive integers whose sum with multiplicity is k . For example, we write $(1, 1, 2)$ for the partition $4 = 1 + 1 + 2$. If $\lambda(j)$ is a partition of $k(j)$, then we denote by $\cup_j \lambda(j)$ the multiset union, which is a partition of $\sum_j k(j)$. A multiset λ_1 is a **submultiset** of λ_2 if for all $r \in \lambda_1$, the multiplicity of occurrence of r in λ_1 is less than or equal to the multiplicity of occurrence of r in λ_2 . A set partition of a finite set \mathbf{A} is a set of nonempty subsets of \mathbf{A} , with pairwise empty intersection, whose union is \mathbf{A} .

Let Λ be a set with some algebraic structure, e.g. a vector space. A **poset-filtration** of Λ is a collection Λ^λ of sub-objects of Λ indexed by elements of a partially ordered set $(\{\lambda\}, \leq)$ with the property that $\Lambda^{\lambda_1} \subseteq \Lambda^{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$.

4. Rational correspondences and dynamical degrees

In this section X , X_1 , X_2 , and X_3 are smooth irreducible projective varieties.

4.1. Rational correspondences

A rational correspondence from X_1 to X_2 is a multivalued map from a dense open set in X_1 to X_2 .

Definition 4.1. A **rational correspondence** $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$ is a diagram

$$\begin{array}{ccc} & \Gamma & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X_1 & & X_2 \end{array}$$

where Γ is a smooth quasiprojective variety, not necessarily irreducible, and the restriction of π_1 to every irreducible component of Γ is dominant and generically finite.

We sometimes suppress part of the notation and write $\Gamma : X_1 \dashrightarrow X_2$ for the rational correspondence $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$.

Over a dense open subset $U_1 \subseteq X_1$, π_1 is a finite covering map of some degree d , so $\pi_2 \circ \pi_1^{-1}$ defines a multivalued map from U_1 to X_2 and induces a regular map from U_1 to $\text{Sym}^d(X_2)$. Outside U_1 , however, the fibers of π_1 could be empty or positive-dimensional, and it may be impossible to extend this to a multivalued map from X_1 to X_2 , respectively a regular map from X_1 to $\text{Sym}^d(X_2)$.

Rational correspondences induce pushforward and pullback maps:

Definition 4.2. Let $\bar{\Gamma}$ be a smooth projective compactification of Γ such that Γ is dense in $\bar{\Gamma}$ and π_1 and π_2 extend to maps $\bar{\pi}_1$ and $\bar{\pi}_2$ defined on $\bar{\Gamma}$. The cycle $(\bar{\pi}_1 \times \bar{\pi}_2)_*([\bar{\Gamma}]) \in H_{2 \dim X_1}(X_1 \times X_2)$ is independent of the choice of compactification $\bar{\Gamma}$, so we denote this cycle by $[\Gamma]$. Set

$$[\Gamma]_* := (\bar{\pi}_2)_* \circ \bar{\pi}_1^* : H_c(X_1) \rightarrow H_c(X_2)$$

and

$$[\Gamma]^* := (\bar{\pi}_1)_* \circ \bar{\pi}_2^* : H^c(X_2) \rightarrow H^c(X_1).$$

$[\Gamma]_*$ and $[\Gamma]^*$ are well-defined and depend only on $[\Gamma]$. In fact, if pr_1 and pr_2 are the projections from $X_1 \times X_2$ to X_1 and X_2 respectively, we have for $\mathbf{a} \in H_c(X_1)$ and $\mathbf{b} \in H^c(X_2)$

$$[\Gamma]_*(\mathbf{a}) = (\text{pr}_2)_*([\Gamma] \smile \text{pr}_1^*(\mathbf{a}))$$

and

$$[\Gamma]^*(\mathbf{b}) = (\text{pr}_1)_*([\Gamma] \smile \text{pr}_2^*(\mathbf{b})),$$

where \smile denotes cup product in $H^*(X_1 \times X_2)$. The cohomology group $H^c(X_j)$ is dual to the homology group $H_c(X_j)$. By the projection formula, $[\Gamma]_*$ and $[\Gamma]^*$ are dual maps.

Suppose $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$ and $(\Gamma', \pi'_2, \pi'_3) : X_2 \dashrightarrow X_3$ are rational correspondences such that the image under π_2 of every irreducible component of Γ intersects the domain of definition of the multivalued function $\pi'_3 \circ (\pi'_2)^{-1}$. The composite $\Gamma' \circ \Gamma$ is a rational correspondence from X_1 to X_3 defined as follows.

Pick dense open sets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that $\pi_2(\pi_1^{-1}(U_1)) \subseteq U_2$, and $\pi_1|_{\pi_1^{-1}(U_1)}$ and $\pi'_2|_{(\pi'_2)^{-1}(U_2)}$ are both covering maps. Set

$$\Gamma' \circ \Gamma := \pi_1^{-1}(U_1) \xrightarrow{\pi_2 \times \pi'_2} (\pi'_2)^{-1}(U_2),$$

together with its given maps to X_1 and X_3 .

Although this definition of $\Gamma' \circ \Gamma$ depends on the choice of open sets U_1 and U_2 , the cycle class $[\Gamma' \circ \Gamma]$ is well-defined; in fact, if $[\Gamma_1] = [\Gamma_2]$ and $[\Gamma'_1] = [\Gamma'_2]$, then $[\Gamma'_1 \circ \Gamma_1] = [\Gamma'_2 \circ \Gamma_2]$ ([7]). It will be convenient to work with a concrete representative $\Gamma' \circ \Gamma$ in its cycle class. However, composition of rational correspondences is only defined up to equivalence of cycle class.

Unfortunately, it is not necessarily true that $[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$ or that $[\Gamma' \circ \Gamma]^* = [\Gamma']^* \circ [\Gamma]^*$. (See Section 4.4 for an example.) Suppose

$$[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_* : H_c(X_1) \rightarrow H_c(X_3).$$

Then we say Γ' and Γ are ***c-homologically composable***. In this case, by duality of pushforward and pullback, Γ' and Γ are also *c*-cohomologically composable:

$$[\Gamma' \circ \Gamma]^* = [\Gamma]^* \circ [\Gamma']^* : H^c(X_3) \rightarrow H^c(X_1).$$

Remark 4.3. There is a theory of ***correspondences***, as distinct from rational correspondences. A correspondence from X_1 to X_2 is a cycle class in $X_1 \times X_2$. Correspondences also induce maps on (co)homology groups, but these maps are functorial under composition ([12]).

4.2. Any rational map is a rational correspondence

Let $g : X_1 \dashrightarrow X_2$ be a rational map. Then $\text{Gr}(g) := \overline{\{(x, g(x))\}} \subseteq X_1 \times X_2$ together with its projections pr_1 and pr_2 to X_1 and X_2 respectively, is a rational correspondence from X_1 to X_2 . The pushforward g_* and the pullback g^* by g have been independently defined in the literature to be $[\text{Gr}(g)]_*$ and $[\text{Gr}(g)]^*$, respectively ([26]). If $g : X_1 \dashrightarrow X_2$ and $g' : X_2 \dashrightarrow X_3$ are rational maps such that the image of g intersects the domain of definition of g' , then the composite $g' \circ g$ is a rational map $X_1 \dashrightarrow X_3$, and $[\text{Gr}(g' \circ g)] = [\text{Gr}(g') \circ \text{Gr}(g)]$. We may thus identify rational maps with the rational correspondences given by their graphs.

If $(\Gamma, \pi_1, \pi_2) : X_1 \dashrightarrow X_2$ is a rational correspondence with π_1 generically one-to-one, then $[\Gamma]$ is the rational correspondence given by the rational map $\pi_2 \circ \pi_1^{-1}$.

4.3. Dynamical degrees

We refer the reader to [26] or [1] for more extended discussions of dynamical degrees of rational maps. Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a rational self-correspondence such that the restriction of π_2 to every irreducible component of Γ is dominant.

Definition 4.4. In this case we say Γ is a ***dominant*** rational self-correspondence.

Set $\Gamma^n := \Gamma \circ \cdots \circ \Gamma$ (n times). Each $[\Gamma^n]^*$ acts on $H^{2k}(X)$, preserving $H^{k,k}(X)$.

Definition 4.5. Pick a norm $\|\cdot\|$ on $H^{k,k}(X)$. The k th dynamical degree Θ_k of Γ is $\lim_{n \rightarrow \infty} \|[\Gamma^n]^*\|^{1/n}$.

This limit exists ([6]), and is independent of the choice of norm ([3]). In [6] and [7], Dinh and Sibony show that the topological entropy of a rational map or rational correspondence is bounded from above by the logarithm of its largest dynamical degree.

Suppose $H^{k,k}(X) = H^{2k}(X)$. Then, since pullback on $H^{2k}(X)$ is dual to pushforward on $H_{2k}(X)$, we can rewrite

$$(\text{kth dynamical degree of } \Gamma) = \lim_{n \rightarrow \infty} \|[\Gamma]_* : H_{2k}(X) \rightarrow H_{2k}(X)\|^{1/n}.$$

If $[\Gamma^n]^* = ([\Gamma]^*)^n$ on $H^{k,k}(X)$ for all n , Γ is called ***k-stable***. A rational correspondence that is *k-stable* for all *k* is called ***algebraically stable***.

For *k-stable* Γ , we can rewrite the dynamical degree Θ_k as $\lim_{n \rightarrow \infty} \|([\Gamma]^*)^n\|^{1/n}$, which is the absolute value of the dominant eigenvalue of $[\Gamma]^*$. The *kth* dynamical degree is hard to compute for rational maps or correspondences that are not *k-stable*, except in a few examples.

4.4. Example: The Cremona involution on \mathbb{P}^2

Let $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational self-map given in coordinates by

$$[\mathbf{x} : \mathbf{y} : \mathbf{z}] \mapsto [\mathbf{yz} : \mathbf{xz} : \mathbf{xy}].$$

The map g is undefined at the coordinate points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Also, g^2 is the identity where defined — the complement of the coordinate lines. Let $\mathfrak{l} \in H^{1,1}(\mathbb{P}^2)$ be the class of a line. Since g is given by degree two polynomials in the coordinates, it is easy to check that a general line pulls back to a conic. So $g^*\mathfrak{l} = 2\mathfrak{l}$ and g^* acts on $H^{1,1}(\mathbb{P}^2)$ via multiplication by 2. On the other hand, $g^2 = \text{Id}_{\mathbb{P}^2}$, so $(g^2)^*$ acts by the identity on $H^{1,1}(\mathbb{P}^2)$. In particular $(g^2)^* \neq (g^*)^2$.

Although g is not 1-stable, its dynamical degree Θ_1 is easily computable. For n odd, $g^n = g$ on \mathbb{P}^2 , and $(g^n)^* = g^*$ is multiplication by 2 on $H^{1,1}(\mathbb{P}^2)$. For n even, $g^n = \text{Id}_{\mathbb{P}^2}$ and $(g^n)^*$ is the identity on $H^{1,1}(\mathbb{P}^2)$. For any norm $\|\cdot\|$, the sequence $\|(g^n)^*\|^{1/n}$ goes to 1 as n goes to ∞ , so $\Theta_1 = 1$.

We can understand the lack of 1-stability of g by examining its graph $\text{Gr}(g) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$. Let the coordinates on the first \mathbb{P}^2 factor be $\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1$, and the coordinates on the second \mathbb{P}^2 factor be $\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2$. Denote by π_1 and π_2 the projections onto the first and second factors, respectively. Then $\text{Gr}(g)$ is given by the equations

$$\mathbf{x}_1\mathbf{x}_2 = \mathbf{y}_1\mathbf{y}_2 = \mathbf{z}_1\mathbf{z}_2.$$

Over the open set $U \subseteq \mathbb{P}^2$ where all coordinates are nonzero, $\text{Gr}(g)$ has equations

$$\mathbf{x}_1 = \frac{1}{\mathbf{x}_2}, \quad \mathbf{y}_1 = \frac{1}{\mathbf{y}_2}, \quad \mathbf{z}_1 = \frac{1}{\mathbf{z}_2},$$

so π_1 and π_2 are equal. The fibered product $V = \text{Gr}(g) \times_{\pi_2 \times \pi_1} \text{Gr}(g)$ embeds naturally in $\mathbb{P}^2 \times \mathbb{P}^2$ and has four irreducible components:

$$\begin{aligned} V_{\text{diag}} &:= \{[\mathbf{x}_1 : \mathbf{y}_1 : \mathbf{z}_1] = [\mathbf{x}_2 : \mathbf{y}_2 : \mathbf{z}_2]\} \\ V_{\mathbf{x}} &:= \{\mathbf{x}_1 = 0\} \times \{\mathbf{x}_2 = 0\} \\ V_{\mathbf{y}} &:= \{\mathbf{y}_1 = 0\} \times \{\mathbf{y}_2 = 0\} \\ V_{\mathbf{z}} &:= \{\mathbf{z}_1 = 0\} \times \{\mathbf{z}_2 = 0\}. \end{aligned}$$

None of $V_{\mathbf{x}}$, $V_{\mathbf{y}}$, or $V_{\mathbf{z}}$ maps dominantly onto either \mathbb{P}^2 factor, so V does not define a rational correspondence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. However, it induces the map $\mathfrak{l} \mapsto 4\mathfrak{l}$ on $H^{1,1}(\mathbb{P}^2)$, which is the same as $(g^*)^2$. On the other hand, the graph of g^2 is V_{diag} , just one of the four irreducible components of V .

4.5. Birationally conjugate rational correspondences

Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational self-correspondence, and let $\rho : X \dashrightarrow X'$ be a birational equivalence. Then we obtain a dominant rational self-correspondence on X' through conjugation by ρ , as follows. Let U be the domain of definition of ρ . Set $\Gamma' = \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$. Then

$$(\Gamma', \rho \circ \pi_1, \rho \circ \pi_2) : X' \dashrightarrow X'$$

is a dominant rational self-correspondence.

Theorem 4.6 ([6, 7, 31]). *The dynamical degrees of Γ and Γ' are equal.*

Remark 4.7. Theorem 4.6 as stated does not appear in the references quoted above. The proofs in [31] were modified in the Appendix to [25] to give a complete proof.

Thus we can study the dynamical degrees of $\Gamma : X \dashrightarrow X$ via the action of Γ on the birational model X' . The next two lemmas allow us to compare a rational correspondence on different birational models.

Lemma 4.8. *Let X and X' be smooth projective varieties, with $H^{2k}(X) = H^{k,k}(X)$ and $H^{2k}(X') = H^{k,k}(X')$ for some k . Suppose $\Gamma : X \dashrightarrow X$ is a k -stable rational self-correspondence, $\Lambda \subseteq H_{2k}(X)$ is a subspace with $[\Gamma]_*(\Lambda) \subseteq \Lambda$, and X' admits a birational morphism $\rho : X \rightarrow X'$ such that $\Lambda \subseteq \ker(\rho_*)$. Set $\Omega = H_{2k}(X)/\Lambda$. Then the k th dynamical degree of Γ is the absolute value of the dominant eigenvalue of the induced map $[\Gamma]_* : \Omega \rightarrow \Omega$.*

Proof. Dynamical degrees are birational invariants (Theorem 4.6), so the k th dynamical degree of Γ on X is equal to its k th dynamical degree on X' . Denote by $[\Gamma^n]_*^X$ the pushforward induced by the n th iterate of Γ on $H_{2k}(X)$, by $[\Gamma^n]_*^\Omega$ the induced map on Ω , and by $[\Gamma^n]_*^{X'}$ the pushforward on $H_{2k}(X')$. By the k -stability of Γ on X , we have

$$\begin{aligned} [\Gamma^n]_*^X &= ([\Gamma]_*^X)^n \\ [\Gamma^n]_*^\Omega &= ([\Gamma]_*^\Omega)^n. \end{aligned}$$

Denote by pr the map from $H_{2k}(X)$ to Ω . Pick norms on $H_{2k}(X)$, Ω , and $H_{2k}(X')$. This induces norms on maps among these vector spaces. We abuse notation by using $\|\cdot\|$ to denote all of these norms. Since $\Lambda \subseteq \ker(\rho_*)$, there is a factorization:

$$\begin{array}{ccc} H_{2k}(X) & \xrightarrow{\rho_*} & H_{2k}(X') \\ & \searrow \text{pr} & \nearrow \overline{\rho}_* \\ & \Omega & \end{array}$$

For $n > 0$, we have

$$\begin{aligned} [\Gamma^n]_*^{X'} &= \rho_* \circ [\Gamma^n]_*^X \circ \rho^* \\ &= \overline{\rho}_* \circ \text{pr} \circ [\Gamma^n]_*^X \circ \rho^* \\ &= \overline{\rho}_* \circ [\Gamma^n]_*^\Omega \circ \text{pr} \circ \rho^* \\ &= \overline{\rho}_* \circ ([\Gamma]_*^\Omega)^n \circ \text{pr} \circ \rho^*. \end{aligned}$$

Thus by submultiplicativity of the induced norms:

$$\|[\Gamma^n]_*^{X'}\| \leq \|\overline{\rho}_*\| \|\text{pr}\| \|\rho^*\| \|([\Gamma]_*^\Omega)^n\|.$$

Taking n th roots and the limit as $n \rightarrow \infty$, we obtain:

$$(k\text{th dynamical degree of } \Gamma) \leq |\text{dominant eigenvalue of } [\Gamma]_*^\Omega|.$$

On the other hand, since Γ is k -stable on X ,

$$\begin{aligned} (k\text{th dynamical degree of } \Gamma) &= |\text{dominant eigenvalue of } [\Gamma]_*^X| \\ &\geq |\text{dominant eigenvalue of } [\Gamma]_*^\Omega|. \end{aligned} \quad \square$$

Lemma 4.9. *Let $\Gamma' : X_2 \dashrightarrow X_3$ be a rational correspondence. For $j \in \{2, 3\}$, let X'_j be a smooth projective variety admitting a birational morphism ρ_j from X_j . Suppose for fixed k , $[\Gamma']_* : H_{2k}(X_2) \rightarrow H_{2k}(X_3)$ takes $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$. Then*

(i) *The following diagram commutes:*

$$\begin{array}{ccc} H_{2k}(X_2) & \xrightarrow{[\Gamma']_*} & H_{2k}(X_3) \\ (\rho_2)_* \downarrow & & \downarrow (\rho_3)_* \\ H_{2k}(X'_2) & \xrightarrow{[\Gamma']_*} & H_{2k}(X'_3) \end{array}$$

Thus $[\Gamma']_* : H_{2k}(X'_2) \rightarrow H_{2k}(X'_3)$ can be identified with the induced map

$$[\Gamma']_* : H_{2k}(X_2) / \ker((\rho_2)_*) \rightarrow H_{2k}(X_3) / \ker((\rho_3)_*),$$

(ii) *If $[\Gamma']_* : H_{2k}(X_2) \rightarrow H_{2k}(X_3)$ takes effective classes to effective classes, then so does $[\Gamma']_* : H_{2k}(X'_2) \rightarrow H_{2k}(X'_3)$, and*

(iii) *If $\Gamma : X_1 \dashrightarrow X_2$ is $2k$ -homologically composable with $\Gamma' : X_2 \dashrightarrow X_3$, then $\Gamma : X_1 \dashrightarrow X'_2$ is $2k$ -homologically composable with $\Gamma' : X'_2 \dashrightarrow X'_3$.*

Proof. Since we discuss the same rational correspondence on different spaces, we include a superscript in the notation for pushforward; e.g. we denote by $[\Gamma']_*^{X_2, X_3}$ the induced map from $H_{2k}(X_2)$ to $H_{2k}(X_3)$, and we denote by $[\Gamma']_*^{X'_2, X'_3}$ the induced map from $H_{2k}(X'_2)$ to $H_{2k}(X'_3)$.

Since the morphism ρ_2 is birational, $(\rho_2)_* \circ \rho_2^*$ is the identity on $H_{2k}(X'_2)$. Thus:

$$\text{im}(\rho_2^* \circ (\rho_2)_* - \text{Id}) \subseteq \ker((\rho_2)_*).$$

By assumption $[\Gamma']_*^{X_2, X_3}$ sends $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$. Thus

$$\begin{aligned} [\Gamma']_*^{X'_2, X'_3} \circ (\rho_2)_* &= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3} \circ \rho_2^* \circ (\rho_2)_* \\ &= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3}. \end{aligned} \quad (1)$$

This proves (i).

For α an effective k -dimensional cycle on X'_2 , there is an effective cycle $\tilde{\alpha}$ on X_2 satisfying $(\rho_2)_*(\tilde{\alpha}) = \alpha$. Thus $\tilde{\alpha} - \rho_2^*(\alpha) \in \ker((\rho_2)_*)$. Again, since $[\Gamma']_*^{X_2, X_3}$ sends $\ker((\rho_2)_*)$ to $\ker((\rho_3)_*)$,

$$\begin{aligned} [\Gamma']_*^{X'_2, X'_3}(\alpha) &= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3} \circ (\rho_2^*)(\alpha) \\ &= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3}(\tilde{\alpha}). \end{aligned}$$

By assumption, $[\Gamma']_*^{X_2, X_3}$ preserves effectiveness. The pushforward by the regular map ρ_3 preserves effective-

ness, so $[\Gamma']_*^{X'_2, X'_3}(\alpha)$ is effective, proving (ii). Finally, for (iii),

$$\begin{aligned}
[\Gamma']_*^{X'_2, X'_3} \circ [\Gamma]_*^{X_1, X'_2} &= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3} \circ \rho_2^* \circ (\rho_2)_* \circ [\Gamma]_*^{X_1, X_2}. \\
&= (\rho_3)_* \circ [\Gamma']_*^{X_2, X_3} \circ [\Gamma]_*^{X_1, X_2} \quad \text{by (1)} \\
&= (\rho_3)_* \circ [\Gamma' \circ \Gamma]_*^{X_1, X_3} \\
&= [\Gamma' \circ \Gamma]_*^{X_1, X'_3}.
\end{aligned}$$

□

5. $\mathcal{M}_{0,N}$ and Hurwitz correspondences

The moduli space $\mathcal{M}_{0,N}$ parametrizes all ways of labeling N distinct points on \mathbb{P}^1 , up to projective change of coordinates.

Definition 5.1. An N -*marked smooth genus zero curve* is a curve C , isomorphic to \mathbb{P}^1 , together with distinct labeled points $p_1, \dots, p_N \in C$. For \mathbf{P} a finite set, an \mathbf{P} -*marked smooth genus zero curve* is a curve C , isomorphic to \mathbb{P}^1 , together with an injective map $\mathbf{P} \hookrightarrow C$.

Definition 5.2. Let $N \geq 3$. There is a smooth quasiprojective variety $\mathcal{M}_{0,N}$ of dimension $N - 3$ parametrizing all N -marked smooth genus zero curves up to isomorphism. Likewise, for \mathbf{P} a finite set with $|\mathbf{P}| \geq 3$, there is a moduli space $\mathcal{M}_{0,\mathbf{P}} \cong \mathcal{M}_{0,|\mathbf{P}|}$, parametrizing smooth genus zero \mathbf{P} -marked curves up to isomorphism.

Definition 5.3. For $N \geq N' \geq 3$, there is a *forgetful map* $\mu : \mathcal{M}_{0,N} \rightarrow \mathcal{M}_{0,N'}$ sending (C, p_1, \dots, p_N) to $(C, p_1, \dots, p_{N'})$. Similarly, for $\mathbf{P}' \hookrightarrow \mathbf{P}$ with $|\mathbf{P}'| \geq 3$, there is a forgetful map $\mu : \mathcal{M}_{0,\mathbf{P}} \rightarrow \mathcal{M}_{0,\mathbf{P}'}$.

Hurwitz spaces are moduli spaces parametrizing finite maps with prescribed ramification between smooth curves of prescribed genus. In this paper, we only deal with Hurwitz spaces of maps between genus zero curves. See [27] for a summary and proofs of facts quoted here.

Definition 5.4 (*Hurwitz space*, [25], Definition 2.2). Fix discrete data:

- \mathbf{A} and \mathbf{B} finite sets with cardinality at least 3 (marked points on source and target curves, respectively),
- d a positive integer (degree),
- $F : \mathbf{A} \rightarrow \mathbf{B}$ a map,
- $\text{br} : \mathbf{B} \rightarrow \{\text{partitions of } d\}$ (branching), and
- $\text{rm} : \mathbf{A} \rightarrow \mathbb{Z}^{>0}$ (ramification),

such that

- (Condition 1, Riemann-Hurwitz constraint) $\sum_{b \in \mathbf{B}} (d - \text{length of } \text{br}(b)) = 2d - 2$, and
- (Condition 2) for all $b \in \mathbf{B}$, the multiset $(\text{rm}(a))_{a \in F^{-1}(b)}$ is a submultiset of $\text{br}(b)$.

There exists a smooth quasiprojective variety $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$, a *Hurwitz space*, parametrizing morphisms $f : C \rightarrow D$ up to isomorphism, where

- C and D are \mathbf{A} -marked and \mathbf{B} -marked smooth connected genus zero curves, respectively,
- f is degree d ,
- for all $a \in \mathbf{A}$, $f(a) = F(a)$ (via the injections $\mathbf{A} \hookrightarrow C$ and $\mathbf{B} \hookrightarrow D$),
- for all $b \in \mathbf{B}$, the branching of f over b is given by the partition $\text{br}(b)$, and
- for all $a \in \mathbf{A}$, the local degree of f at a is equal to $\text{rm}(a)$.

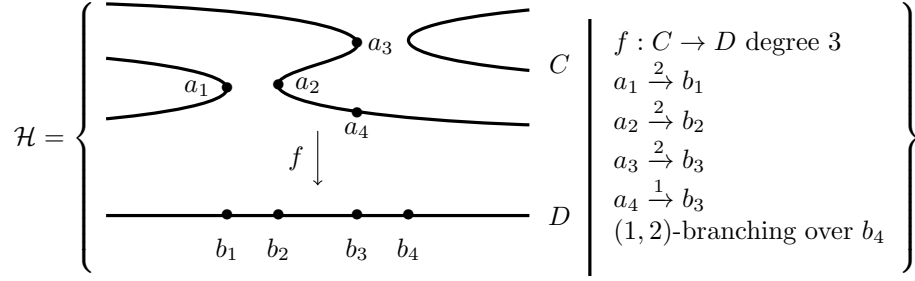


Figure 1: The Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$, where

- $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$,
- $\mathbf{B} = \{b_1, b_2, b_3, b_4\}$,
- $d = 3$,
- $F(a_1) = b_1, F(a_2) = b_2, F(a_3) = F(a_4) = b_3$,
- $\text{br}(b_i) = (1, 2)$ for $i = 1, \dots, 4$, and
- $\text{rm}(a_1) = \text{rm}(a_2) = \text{rm}(a_3) = 2, \text{rm}(a_4) = 1$.

Remark 5.5. One may construct \mathcal{H} as follows ([16]). Consider $\mathcal{M}_{0,\mathbf{A}} \times \mathcal{M}_{0,\mathbf{B}}$ with its two universal curves $\mathcal{U}_{0,\mathbf{A}}$ and $\mathcal{U}_{0,\mathbf{B}}$. Let Hilb be the relative Hilbert scheme of degree d morphisms $\mathcal{U}_{0,\mathbf{A}} \rightarrow \mathcal{U}_{0,\mathbf{B}}$. The conditions that $a \in \mathbf{A}$ map to $F(a)$ with local degree $\text{rm}(a)$ and that the branching over $b \in \mathbf{B}$ be given by $\text{br}(b)$ are locally closed. Thus \mathcal{H} is a locally closed subvariety of Hilb .

The Hurwitz space \mathcal{H} admits a map $\pi_{\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{A}}$, sending $[f : C \rightarrow D]$ to the marked source curve $[C]$. It similarly admits a map $\pi_{\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{B}}$, sending $[f : C \rightarrow D]$ to the marked target curve $[D]$. The space \mathcal{H} may be empty; if not, the “target curve” map $\pi_{\mathbf{B}}$ is a finite covering map and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{B}}^{-1}$ defines a multivalued map from $\mathcal{M}_{0,\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{B}}$. If $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ are projective compactifications of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$ respectively, $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}}$ is a rational correspondence. We generalize this as follows.

Definition 5.6 (*Hurwitz correspondence*, [25], Definition 2.19). Let \mathbf{A}' be any subset of \mathbf{A} with cardinality at least 3. There is a forgetful morphism $\mu : \mathcal{M}_{0,\mathbf{A}} \rightarrow \mathcal{M}_{0,\mathbf{A}'}$. Let Γ be a union of connected components of \mathcal{H} . If $X_{\mathbf{A}'}$ and $X_{\mathbf{B}}$ are smooth projective compactifications of $\mathcal{M}_{0,\mathbf{A}'}$ and $\mathcal{M}_{0,\mathbf{B}}$ respectively, then

$$(\Gamma, \pi_{\mathbf{B}}, \mu \circ \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}'}$$

is a rational correspondence. We call such a rational correspondence a **Hurwitz correspondence**.

5.1. Connection to dynamics on \mathbb{P}^1 and the Thurston pullback map

This section is purely for motivation. We summarize parts of the results in [19]. Let S^2 denote an oriented 2-sphere.

Definition 5.7. Let $\mathbf{P} \subset S^2$ be a finite set. Define an equivalence relation on orientation-preserving homeomorphisms $S^2 \rightarrow \mathbb{P}^1$ as follows: ψ_1 is equivalent to ψ_2 if there exists $\xi \in \text{Aut}(\mathbb{P}^1)$ such that ψ_1 and $\xi \circ \psi_2$ agree on \mathbf{P} and are isotopic relative to \mathbf{P} . The **Teichmüller space** $\mathcal{T}(S^2, \mathbf{P})$ of (S^2, \mathbf{P}) is the set of equivalence classes of homeomorphisms.

Teichmüller space has a natural structure as a noncompact nonalgebraic complex manifold; it is isomorphic to a bounded domain in $\mathbb{C}^{|\mathbf{P}|-3}$. Given an element $[\psi] \in \mathcal{T}(S^2, \mathbf{P})$, the restriction $\psi|_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbb{P}^1$ defines an element of $\mathcal{M}_{0,\mathbf{P}}$. This gives rise to a map of complex manifolds

$$\mathcal{T}(S^2, \mathbf{P}) \xrightarrow{\text{cv}} \mathcal{M}_{0,\mathbf{P}}.$$

In fact, this is a covering map, realizing $\mathcal{T}(S^2, \mathbf{P})$ as the universal cover of $\mathcal{M}_{0,\mathbf{P}}$.

Definition 5.8. An orientation-preserving branched covering ϕ from S^2 to itself is called *postcritically finite* if the post-critical set

$$\mathbf{P} := \{\phi^n(x) | n > 0, x \text{ a critical point of } \phi\}$$

is finite.

We say ϕ is *combinatorially equivalent* to an algebraic morphism if there exist orientation-preserving homeomorphisms $\psi_1, \psi_2 : S^2 \rightarrow \mathbb{P}^1$ such that

$$\psi_2 \circ \phi \circ \psi_1^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

is an algebraic morphism, and ψ_1 and ψ_2 are isotopic relative to the postcritical set \mathbf{P} . W. Thurston gave a topological characterization for ϕ to be combinatorially equivalent to an algebraic morphism, in terms of curve systems on $S^2 \setminus \mathbf{P}$. This characterization can also be stated in terms of a self-map on $\mathcal{T}(S^2, \mathbf{P})$ ([8]). If $\psi : (S^2, \mathbf{P}) \rightarrow (\mathbb{P}^1, \psi(\mathbf{P}))$ is an orientation-preserving homeomorphism then the induced complex structure on $S^2 \setminus \mathbf{P}$ can be pulled back via the covering map $\phi|_{S^2 \setminus \phi^{-1}(\mathbf{P})}$ to obtain another complex structure on $S^2 \setminus \phi^{-1}(\mathbf{P})$. This extends to a complex structure on all of S^2 , and yields another homeomorphism $\psi' : (S^2, \mathbf{P}) \rightarrow (\mathbb{P}^1, \psi'(\mathbf{P}))$ such that

$$\psi \circ \phi \circ (\psi')^{-1} : (\mathbb{P}^1, \psi'(\mathbf{P})) \rightarrow (\mathbb{P}^1, \psi(\mathbf{P}))$$

is an algebraic morphism. The map

$$\begin{aligned} \mathcal{T}(S^2, \mathbf{P}) &\xrightarrow{\text{Thurst}(\phi)} \mathcal{T}(S^2, \mathbf{P}) \\ [\psi] &\mapsto [\psi'] \end{aligned}$$

is a well-defined holomorphic map ([8]) called the *Thurston pullback map*.

This pullback map is a contraction in the hyperbolic metric on $\mathcal{T}(S^2, \mathbf{P})$; it is a theorem of Thurston that ϕ is combinatorially equivalent to an algebraic map if and only if $\text{Thurst}(\phi)$ has a fixed point. The dynamics of $\text{Thurst}(\phi)$ are thus of interest.

The Thurston pullback map is holomorphic, but not algebraic. However, Koch ([19]) showed that it descends to a Hurwitz correspondence on the algebraic variety $\mathcal{M}_{0,\mathbf{P}}$.

Given $\phi : (S^2, \mathbf{P}) \rightarrow (S^2, \mathbf{P})$, denote by d the topological degree of ϕ . Define

$$\begin{aligned} \text{br} : \mathbf{P} &\rightarrow \{\text{partitions of } d\} \\ p &\mapsto \text{branching of } \phi \text{ over } p \end{aligned}$$

and

$$\begin{aligned} \text{rm} : \mathbf{P} &\rightarrow \mathbb{Z}^{>0} \\ p &\mapsto \text{local degree of } \phi \text{ at } p. \end{aligned}$$

The data $(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbf{P}, \text{br}, \text{rm})$ satisfy Conditions 1 and 2 in Definition 5.4. Denote by \mathcal{H} the Hurwitz space $\mathcal{H}(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbf{P}, \text{br}, \text{rm})$. Given a homeomorphism $\psi : (S^2, \mathbf{P}) \rightarrow (\mathbb{P}^1, \psi(\mathbf{P}))$, there

exists a homeomorphism $\psi' : (S^2, \mathbf{P}) \rightarrow (\mathbb{P}^1, \psi'(\mathbf{P}))$, with $[\psi'] = \text{Thurst}(\phi)([\psi])$, and such that

$$\psi \circ \phi \circ (\psi')^{-1} : (\mathbb{P}^1, \psi'(\mathbf{P})) \rightarrow (\mathbb{P}^1, \psi(\mathbf{P}))$$

is an algebraic morphism. This defines a point in \mathcal{H} . By [19], we obtain a holomorphic covering map

$$\begin{aligned} \mathcal{T}(S^2, \mathbf{P}) &\rightarrow \mathcal{H} \\ [\psi] &\mapsto [(\mathbb{P}^1, \psi'(\mathbf{P})) \xrightarrow{\psi \circ \phi \circ (\psi')^{-1}} (\mathbb{P}^1, \psi(\mathbf{P}))] \end{aligned}$$

whose image is a connected component Γ of \mathcal{H} . We have the commutative diagram:

$$\begin{array}{ccccc} \mathcal{T}(S^2, \mathbf{P}) & \xrightarrow{\text{Thurst}(\phi)} & \mathcal{T}(S^2, \mathbf{P}) & & \\ \downarrow \text{cv} & \searrow & \downarrow \text{cv} & & \\ & \Gamma & & & \\ \swarrow \pi_{\mathbf{P}}^{\text{target}} & & \searrow \pi_{\mathbf{P}}^{\text{source}} & & \\ \mathcal{M}_{0, \mathbf{P}} & & \mathcal{M}_{0, \mathbf{P}} & & \end{array}$$

Thus the Hurwitz correspondence Γ is an algebraic “shadow” of the nonalgebraic Thurston pullback map. Our study of the dynamics of Hurwitz self-correspondences is motivated in part by this connection to the dynamics of the Thurston pullback map.

6. Compactifications of $\mathcal{M}_{0, N}$

$\mathcal{M}_{0, N}$ parametrizes smooth curves of genus zero with N distinct marked points. However, when $N > 3$ there are 1-parameter families $C(\mathbf{t})_{\mathbf{t} \neq 0}$ of smooth curves with N distinct marked points such that as \mathbf{t} goes to zero, there is no limiting smooth curve where the marked points remain distinct. Thus $\mathcal{M}_{0, N}$ cannot be compact. There are many compactifications of $\mathcal{M}_{0, N}$ that are essentially based on describing what happens when marked points collide.

$\overline{\mathcal{M}}_{0, N}$ is the most widely studied of these compactifications. Here, the marked points are always distinct, but the curve may be nodal. Moduli spaces of weighted stable curves are a generalization of $\overline{\mathcal{M}}_{0, N}$ constructed by Hassett in [17]. Here, marked points are assigned weights between 0 and 1, and a set of marked points may coincide as long as their total weight is not more than 1.

6.1. $\overline{\mathcal{M}}_{0, N}$ and its combinatorial structure

We refer the reader to [21] for an extended discussion.

Definition 6.1. A *stable N -marked genus zero curve* is a connected algebraic curve C of arithmetic genus zero whose only singularities are simple nodes, together with N distinct smooth marked points p_1, \dots, p_N on C , such that the set of automorphisms $C \rightarrow C$ that fix every marked point p_i is finite.

The irreducible components of such C are all isomorphic to \mathbb{P}^1 . Points of C that are either marked points or nodes are called *special points*. The *stability* condition that (C, p_1, \dots, p_N) has finitely many automorphisms implies that it has no nontrivial automorphisms, and is equivalent to the condition that every irreducible component of C has at least three special points.

Theorem 6.2 (Deligne-Mumford, Grothendieck, Knudsen). *Let $N \geq 3$. There is a smooth projective variety $\overline{\mathcal{M}}_{0, N}$ of dimension $N - 3$ that is a fine moduli space for stable N -marked genus zero curves. It contains $\mathcal{M}_{0, N}$ as a dense open subset.*

For \mathbf{P} a finite set, we analogously define stable genus zero \mathbf{P} -marked curves, and their moduli space $\overline{\mathcal{M}}_{0,\mathbf{P}}$ (isomorphic to $\overline{\mathcal{M}}_{0,|\mathbf{P}|}$). It contains $\mathcal{M}_{0,\mathbf{P}}$ as a dense open subset.

The **boundary** $\overline{\mathcal{M}}_{0,N} \setminus \mathcal{M}_{0,N}$ is a simple normal crossings divisor. Points on the boundary correspond to reducible stable curves. The topological type of a stable curve is captured by the combinatorial information of its dual tree. This classification of stable curves by topological type gives a stratification of $\overline{\mathcal{M}}_{0,N}$.

Definition 6.3. Let (C, p_1, \dots, p_N) be a stable genus zero curve. Its **dual tree** σ is the graph defined as follows. The vertices v of σ correspond to the irreducible components C_v of C . Two vertices v_1 and v_2 are connected by an edge if and only if the components C_{v_1} and C_{v_2} meet at a node. For each marked point p_i on C_v , we attach a **leg** ℓ_i to the vertex v . Note that edges and legs are distinct from each other. The graph σ is a tree because C has arithmetic genus zero.

For a vertex v on σ , set

$$\Delta_v = \{\text{Legs attached to } v\} \cup \{\text{edges incident to } v\}.$$

We refer to elements of Δ_v as **flags** on v . We define the **valence** of v , denoted $|v|$, to be the cardinality of Δ_v . For $i \in \{1, \dots, N\}$, we define $\delta(v \rightarrow i)$ to be the unique flag in Δ_v that connects the leg ℓ_i to v , i.e. is part of the unique non-repeating path in σ from v to ℓ_i . If $\ell_i \in \Delta_v$, then $\delta(v \rightarrow i) = \ell_i$; otherwise $\delta(v \rightarrow i)$ is an edge. There is a canonical injection $\Delta_v \hookrightarrow C_v$ whose image is the set of special points of C_v . Thus C_v is a Δ_v -marked smooth genus zero curve. We denote by $\mathcal{V}(\sigma)$ the set of vertices of σ . We define the **moduli dimension** $\text{md}(v)$ of $v \in \mathcal{V}(\sigma)$ to be $|v| - 3$.

Definition 6.4. A **stable N -marked tree** is a tree σ with marked legs ℓ_1, \dots, ℓ_N such that every vertex has valence at least 3.

For fixed N , there are finitely many isomorphism classes of stable N -marked trees, and each of these arises as the dual tree of some stable N -marked genus zero curve.

Definition 6.5. Given σ a stable N -marked tree, the closure S_σ of the set $\{(C, p_1, \dots, p_N) : C \text{ has dual graph } \sigma\}$ is an irreducible subvariety of $\overline{\mathcal{M}}_{0,N}$. These special subvarieties are called **boundary strata**.

Boundary strata on $\overline{\mathcal{M}}_{0,N}$ are in bijection with isomorphism classes of stable N -marked trees.

Boundary strata on $\overline{\mathcal{M}}_{0,N}$ decompose into products of smaller-dimensional spaces of stable curves. Let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,N}$. Given stable curves $([C_v] \in \overline{\mathcal{M}}_{0,\Delta_v})_{v \in \mathcal{V}(\sigma)}$, we can glue the curves C_v together to obtain a curve C in S_σ as follows. Whenever there is an edge between v_1 and v_2 , glue C_{v_1} to C_{v_2} at the corresponding marked point of each curve. This procedure defines a **gluing morphism**

$$\prod_{v \in \mathcal{V}(\sigma)} \overline{\mathcal{M}}_{0,\Delta_v} \cong S_\sigma \hookrightarrow \overline{\mathcal{M}}_{0,N}.$$

So

$$\dim S_\sigma = \sum_{v \in \mathcal{V}(\sigma)} (|v| - 3) = \sum_{v \in \mathcal{V}(\sigma)} \text{md}(v).$$

The codimension of S_σ in $\overline{\mathcal{M}}_{0,N}$ equals the number of edges of σ .

Forgetful maps. For $N \geq N' \geq 3$, the forgetful map $\mu : \mathcal{M}_{0,N} \rightarrow \mathcal{M}_{0,N'}$ extends to $\mu : \overline{\mathcal{M}}_{0,N} \rightarrow \overline{\mathcal{M}}_{0,N'}$. Let (C, p_1, \dots, p_N) be any N -marked stable curve. The curve $(C, p_1, \dots, p_{N'})$ obtained by forgetting $p_{N'+1}, \dots, p_N$ may not be stable. However, we obtain $(C', p_1, \dots, p_{N'})$ a stable N' -marked curve by *stabilizing*, i.e. successively contracting components of C with fewer than 3 special points. The map μ sends $[(C, p_1, \dots, p_N)]$ to $[(C', p_1, \dots, p_{N'})]$.

If σ is the dual tree of C , we obtain the dual tree σ' of C' by deleting legs $\ell_{N'+1}, \dots, \ell_N$ and applying a finite sequence of steps, also called stabilization. Each step is either of the form:

- For a vertex v of valence 1, delete v , together with its incident edge, or
- For a vertex v of valence 2 with edges e_1 and e_2 connecting v to vertices v_1 and v_2 , respectively, delete v together with e_1 and e_2 , and connect v_1 to v_2 by an edge, or
- For a vertex v of valence 2 incident to an edge e connecting v to another vertex v_1 , and also incident to a leg ℓ_i , delete v together with e , and attach ℓ_i to v_1 .

The deletion of the vertex v on σ corresponds to the contraction of C_v on C .

Homology groups of $\overline{\mathcal{M}}_{0,N}$. The homology group $H_{2k}(\overline{\mathcal{M}}_{0,N})$ is isomorphic to the Chow group $A_k(\overline{\mathcal{M}}_{0,N})$ and is generated by the classes of k -dimensional boundary strata. $\overline{\mathcal{M}}_{0,N}$ has no odd-dimensional homology (Keel, [18]).

Additive relations among boundary strata are described in [22] by Kontsevich and Manin. Let σ be the dual tree of some $(k+1)$ -dimensional boundary stratum, let v be a vertex on σ with valence at least four, and let $i_1, i_2, i_3, i_4 \in \{1, \dots, N\}$ be such that the flags $\delta(v \rightarrow i_1), \dots, \delta(v \rightarrow i_4)$ on v are distinct. The data $\sigma, v, i_1, \dots, i_4$ determine a relation $R(\sigma, v, i_1, \dots, i_4)$ among k -dimensional boundary strata. For every set partition $\Delta_v \setminus \{\delta(v \rightarrow i_1), \dots, \delta(v \rightarrow i_4)\} = \Delta_1 \sqcup \Delta_2$, define a stable tree $\sigma(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)$ as follows.

1. Insert an edge into v , splitting it into two vertices v_1 and v_2 .
2. Attach the flags in $\Delta_1 \cup \{\delta(v \rightarrow i_1), \delta(v \rightarrow i_2)\}$ to v_1 .
3. Attach the flags in $\Delta_2 \cup \{\delta(v \rightarrow i_3), \delta(v \rightarrow i_4)\}$ to v_2 .

The tree $\sigma(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)$ has one more edge than σ and thus corresponds to a k -dimensional boundary stratum $S(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)$. The tree $\sigma(i_1 i_3 \Delta_1 | \Delta_2 i_2 i_4)$ and stratum $S(i_1 i_3 \Delta_1 | \Delta_2 i_2 i_4)$ are defined analogously. Then

$$\begin{aligned} R(\sigma, v, i_1, \dots, i_4) &:= \sum_{(\Delta_1, \Delta_2)} [S(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)] - \sum_{(\Delta_1, \Delta_2)} [S(i_1 i_3 \Delta_1 | \Delta_2 i_2 i_4)] \\ &= 0 \in H_{2k}(\overline{\mathcal{M}}_{0,N}) \quad (\text{resp. } A_k(\overline{\mathcal{M}}_{0,N})), \end{aligned} \tag{2}$$

where the sum is over set partitions $\Delta_v \setminus \{\delta(v \rightarrow i_1), \dots, \delta(v \rightarrow i_4)\} = \Delta_1 \cup \Delta_2$. These relations, varying $\sigma, v, i_1, \dots, i_4$, generate all additive relations among k -dimensional boundary strata.

6.2. Weighted stable curves

Definition 6.6 (Hassett, [17]). A **weight datum** is a tuple $\epsilon = (\epsilon_1, \dots, \epsilon_N) \in (\mathbb{Q} \cap (0, 1])^N$ such that $\sum_{i=1}^N \epsilon_i > 2$.

Definition 6.7 (Hassett, [17]). Let ϵ be a weight datum. A **genus zero ϵ -stable curve** is a connected algebraic curve C of arithmetic genus zero, whose only singularities are simple nodes, together with smooth marked points p_1, \dots, p_N , not necessarily distinct, such that

- (1) If $p_{i_1} = \dots = p_{i_s}$ then $\epsilon_{i_1} + \dots + \epsilon_{i_s} \leq 1$, and
- (2) For any irreducible component C_v ,

$$\#\{\text{nodes on } C_v\} + \sum_{p_i \text{ on } C_v} \epsilon_i > 2.$$

Definition-Theorem 6.8 (Hassett, [17]). Given a weight datum ϵ , there is a smooth projective variety $\overline{\mathcal{M}}_{0,N}(\epsilon)$ of dimension $N-3$ that is a fine moduli space for ϵ -stable genus zero curves. It contains $\mathcal{M}_{0,N}$ as a dense open set. There is a **reduction morphism** $\rho_\epsilon : \overline{\mathcal{M}}_{0,N} \rightarrow \overline{\mathcal{M}}_{0,N}(\epsilon)$ that respects the open inclusion of $\mathcal{M}_{0,N}$ into both spaces.

Definition 6.9. A *boundary stratum* in $\overline{\mathcal{M}}_{0,N}(\epsilon)$ is the image, under ρ_ϵ , of a boundary stratum in $\overline{\mathcal{M}}_{0,N}$.

The homology group $H_{2k}(\overline{\mathcal{M}}_{0,N}(\epsilon))$ is isomorphic to the Chow group $A_k(\overline{\mathcal{M}}_{0,N}(\epsilon))$ and is generated by the classes of boundary strata ([5]).

Remark 6.10. $\overline{\mathcal{M}}_{0,N} = \overline{\mathcal{M}}_{0,N}(\epsilon)$ for $\epsilon = (1, \dots, 1)$.

Definition 6.11. Let ϵ be a weight datum, and σ be an N -marked stable tree. A vertex v on σ is called ϵ -stable if

$$\sum_{\delta \in \Delta_v} \min \left\{ 1, \sum_{i | \delta = \delta(v \rightarrow i)} \epsilon_i \right\} > 2.$$

For $[C, p_1, \dots, p_N] \in \overline{\mathcal{M}}_{0,N}$ with dual tree σ , an ϵ -stable curve representing $\rho_\epsilon([C])$ is obtained by contracting to a point every irreducible component C_v corresponding to $v \in \mathcal{V}(\sigma)$ that is not ϵ -stable. It follows that σ has at least one ϵ -stable vertex. Also, the image under ρ_ϵ of the boundary stratum S_σ has dimension

$$\sum_{\substack{v \in \mathcal{V}(\sigma) \\ v \text{ } \epsilon\text{-stable}}} \text{md}(v).$$

Thus, we obtain:

Lemma 6.12. Let S_σ be a boundary stratum in $\overline{\mathcal{M}}_{0,N}$. The pushforward $(\rho_\epsilon)_*([S_\sigma])$ is nonzero if and only if every vertex $v \in \mathcal{V}(\sigma)$ with positive moduli dimension is ϵ -stable.

7. Admissible covers

There is a widely used compactification of the Hurwitz space \mathcal{H} by a space $\overline{\mathcal{H}}$ of *admissible covers*. Moduli spaces of admissible covers were constructed in [16] by Harris and Mumford, and parametrize finite maps between possibly nodal curves. In general, they are only coarse moduli spaces, with quotient singularities. For technical ease, we introduce a class of Hurwitz spaces whose admissible covers compactifications are fine moduli spaces. We refer to Hurwitz spaces in this class as *fully marked*.

7.1. Fully marked Hurwitz spaces

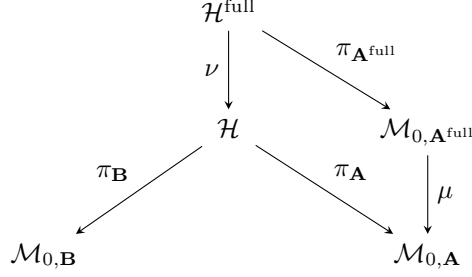
Definition 7.1. Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as in Definition 5.4 with Condition 2 strengthened to:

- (Condition 2') For all $b \in \mathbf{B}$, the multiset $(\text{rm}(a))_{a \in F^{-1}(b)}$ is **equal to** $\text{br}(b)$,

we refer to $\mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as a *fully marked Hurwitz space*.

Given any Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$, we can construct a fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ with a finite covering map $\nu : \mathcal{H}^{\text{full}} \rightarrow \mathcal{H}$ as follows. We first construct a superset \mathbf{A}^{full} of \mathbf{A} , extending the functions F and rm . For every $b \in \mathbf{B}$, and for every r in the multiset complement $\text{br}(b) \setminus (\text{rm}(a))_{a \in F^{-1}(b) \cap \mathbf{A}}$, add an element $a(b, r)$ to \mathbf{A}^{full} , set $F(a(b, r)) = b$, and set $\text{rm}(a(b, r)) = r$. The data $(\mathbf{A}^{\text{full}}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfy Conditions 1 and 2', so $\mathcal{H}^{\text{full}} = \mathcal{H}(\mathbf{A}^{\text{full}}, \mathbf{B}, d, F, \text{br}, \text{rm})$ is a fully marked Hurwitz space.

Let $\text{Aut}(\mathbf{A}^{\text{full}} \setminus \mathbf{A})$ be the subgroup of permutations of $\mathbf{A}^{\text{full}} \setminus \mathbf{A}$ preserving the functions F and rm . This automorphism group acts freely on $\mathcal{H}^{\text{full}}$ by relabeling points in $\mathbf{A}^{\text{full}} \setminus \mathbf{A}$, and the quotient of this action is \mathcal{H} . Denote by ν the quotient map $\mathcal{H}^{\text{full}} \rightarrow \mathcal{H}$. The fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ admits a map $\pi_{\mathbf{A}^{\text{full}}}$ to $\mathcal{M}_{0, \mathbf{A}^{\text{full}}}$. Also, the injection $\mathbf{A} \hookrightarrow \mathbf{A}^{\text{full}}$ yields a forgetful map $\mu : \mathcal{M}_{0, \mathbf{A}^{\text{full}}} \rightarrow \mathcal{M}_{0, \mathbf{A}}$. The following diagrams commutes:



For Γ any union of connected components of \mathcal{H} , $\Gamma^{\text{full}} := \nu^{-1}(\Gamma)$ is a union of connected components of $\mathcal{H}^{\text{full}}$. For compactifications $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$ respectively, $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}})$ and $(\Gamma^{\text{full}}, \pi_{\mathbf{B}} \circ \nu, \mu \circ \pi_{\mathbf{A}^{\text{full}}})$ are both Hurwitz correspondences from $X_{\mathbf{B}}$ to $X_{\mathbf{A}}$, and $[\Gamma^{\text{full}}] = (\deg \nu)[\Gamma]$ in $H_*(X_{\mathbf{B}} \times X_{\mathbf{A}})$. This observation yields a useful lemma:

Lemma 7.2. *Let $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \rightarrow X_{\mathbf{A}}$ be any Hurwitz correspondence. Then*

$$[\Gamma] = \frac{1}{\deg \nu} [\Gamma^{\text{full}}],$$

where each Γ^{full} is a union of connected components of a fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ corresponding to a superset \mathbf{A}^{full} of \mathbf{A} , and $\nu : \Gamma^{\text{full}} \rightarrow \Gamma$ is a finite covering map.

This means that any Hurwitz correspondence can be written in terms of (connected components of) fully marked Hurwitz spaces. These have convenient compactifications by moduli spaces of admissible covers. We use this Lemma in Proposition 8.1 and Theorem 9.6.

7.2. Admissible Covers

For an introduction see [15], Chapter 3G.

Definition 7.3 ([16]). Fix $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as in Definition 5.4. An $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ -admissible cover is a map of curves $f : C \rightarrow D$, where

- (1) D is a \mathbf{B} -marked genus zero stable curve,
- (2) C is a (not necessarily stable) connected nodal genus zero curve, with an injection from \mathbf{A} into the smooth locus of C ,
- (3) $f : C \rightarrow D$ is a finite map of degree d , such that
 - for all $a \in \mathbf{A}$, $f(a) = F(a)$ (via the injections $\mathbf{A} \hookrightarrow C$ and $\mathbf{B} \hookrightarrow D$),
 - for all $b \in \mathbf{B}$, the branching of f over b is given by the partition $\text{br}(b)$,
 - for all $a \in \mathbf{A}$, the local degree of f at a is equal to $\text{rm}(a)$,
 - η is a node on C if and only if $f(\eta)$ is a node on D ,
 - (balancing condition) for η a node of C between irreducible components C_1 and C_2 of C , $f(C_1)$ and $f(C_2)$ are distinct components of D , and the local degree of $f|_{C_1}$ at η is equal to the local degree of $f|_{C_2}$ at η .

Remark 7.4. If $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfies Conditions 1 and 2' as in Definition 7.1, then C is an \mathbf{A} -marked stable curve.

Theorem 7.5 ([16]). *Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfying Conditions 1 and 2' as in Definition 7.1, there is a projective variety $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ that is a fine moduli space for $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ -admissible covers, and contains \mathcal{H} as a dense open subset. $\overline{\mathcal{H}}$ extends the maps $\pi_{\mathbf{A}}$ and $\pi_{\mathbf{B}}$ to maps $\overline{\pi}_{\mathbf{A}}$ and $\overline{\pi}_{\mathbf{B}}$ to $\overline{\mathcal{M}}_{0,\mathbf{A}}$ and $\overline{\mathcal{M}}_{0,\mathbf{B}}$, respectively, with $\overline{\pi}_{\mathbf{B}} : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{B}}$ a finite flat map. $\overline{\mathcal{H}}$ may not be normal, but its normalization is smooth.*

Remark 7.6. The irreducible components of $\overline{\mathcal{H}}$ correspond to the connected components of \mathcal{H} .

Remark 7.7. The construction in [16] is very general, but as stated allows for only simple ramification and no marked points on the source curve, so does not apply to our case. However, it is easily modified: consider $\overline{\mathcal{M}}_{0,\mathbf{A}} \times \overline{\mathcal{M}}_{0,\mathbf{B}}$ with its two universal curves $\overline{\mathcal{U}}_{0,\mathbf{A}}$ and $\overline{\mathcal{U}}_{0,\mathbf{B}}$. Let Hilb be the relative Hilbert scheme of degree d morphisms $\overline{\mathcal{U}}_{0,\mathbf{A}} \rightarrow \overline{\mathcal{U}}_{0,\mathbf{B}}$. Then the locus

$$\overline{\mathcal{H}} := \{f : C \rightarrow D \mid f \text{ is an } (\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})\text{-admissible cover}\}$$

is a closed subscheme of Hilb by the proof of Theorem 4 in [16].

Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfying Conditions 1 and 2 as in Definition 5.4, but not 2' as in Definition 7.1, there is still a compactification $\overline{\mathcal{H}}$ of $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ by admissible covers. Consider the corresponding fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ and its admissible covers compactification $\overline{\mathcal{H}}^{\text{full}}$. The action of $\text{Aut}(\mathbf{A}^{\text{full}} \setminus \mathbf{A})$ on $\mathcal{H}^{\text{full}}$ extends to an action on $\overline{\mathcal{H}}^{\text{full}}$, but this action is no longer free, so the quotient $\overline{\mathcal{H}}$ has *orbifold singularities*, and is only a coarse moduli space.

7.3. Stratification of $\overline{\mathcal{H}}$

Moduli spaces of admissible covers have a stratification analogous to and compatible with that of $\overline{\mathcal{M}}_{0,N}$. In this section we fix $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfying Conditions 1 and 2' as in Definition 7.1. Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ be the corresponding fully marked Hurwitz space, and let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$.

Definition 7.8. Given any admissible cover $[f : C \rightarrow D] \in \overline{\mathcal{H}}$, we can extract its **combinatorial type** $\gamma = (\sigma, \tau, d^{\text{vert}}, f^{\text{vert}}, (F_v)_{v \in \mathcal{V}(\sigma)}, (\text{br}_v)_{v \in \mathcal{V}(\sigma)}, (\text{rm}_v)_{v \in \mathcal{V}(\sigma)})$, where:

- (1) σ is the dual tree of C ,
- (2) τ is the dual tree of D ,
- (3) $d^{\text{vert}} : \mathcal{V}(\sigma) \rightarrow \mathbb{Z}^{>0}$ sends v to $\deg f|_{C_v}$,
- (4) $f^{\text{vert}} : \mathcal{V}(\sigma) \rightarrow \mathcal{V}(\tau)$, where for C_v an irreducible component of C mapping under f to D_w an irreducible component of D , f^{vert} sends v to w ,
- (5) For $v \in \mathcal{V}(\sigma)$, the map $F_v : \Delta_v \rightarrow \Delta_{f^{\text{vert}}(v)}$ is given by the restriction of f via the inclusions $\Delta_v \hookrightarrow C_v$ and $\Delta_{f^{\text{vert}}(v)} \hookrightarrow D_{f^{\text{vert}}(v)}$,
- (6) For $v \in \mathcal{V}(\sigma)$, the map $\text{br}_v : \Delta_{f^{\text{vert}}(v)} \rightarrow \{\text{partitions of } d^{\text{vert}}(v)\}$ sends $\delta' \in \Delta_{f^{\text{vert}}(v)}$ to the branching of $f|_{C_v} : C_v \rightarrow D_{f^{\text{vert}}(v)}$ over δ' , via the inclusion $\Delta_{f^{\text{vert}}(v)} \hookrightarrow D_{f^{\text{vert}}(v)}$, and
- (7) For $v \in \mathcal{V}(\sigma)$, the map $\text{rm}_v : \Delta_v \rightarrow \mathbb{Z}^{>0}$ sends $\delta \in \Delta_v$ to the local degree of $f|_{C_v}$ at δ , via the inclusion $\Delta_v \hookrightarrow C_v$.

Definition 7.9. The closure G_γ of $\{f' : C' \rightarrow D' \mid f' \text{ has combinatorial type } \gamma\}$ is a subvariety of $\overline{\mathcal{H}}$. We call such a subvariety a **boundary stratum** of $\overline{\mathcal{H}}$.

The boundary stratum G_γ in $\overline{\mathcal{H}}$ can be decomposed into a product of lower-dimensional spaces of admissible covers. For $v \in \mathcal{V}(\sigma)$, the data $(\Delta_v, \Delta_{f^{\text{vert}}(v)}, d^{\text{vert}}(v), F_v, \text{br}_v, \text{rm}_v)$ satisfy Conditions 1 and 2' as in Definition 7.1. Denote by $\overline{\mathcal{H}}_v$ the corresponding space of admissible covers. The space $\overline{\mathcal{H}}_v$ admits maps to

the moduli space $\overline{\mathcal{M}}_{0,\Delta_v}$ of source curves and the moduli space $\overline{\mathcal{M}}_{0,\Delta_{f^{\text{vert}}(v)}}$ of target curves. For $w \in \mathcal{V}(\tau)$, set $\overline{\mathcal{H}}_w$ to be the *fibered* product

$$\overline{\mathcal{H}}_w := \prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{H}}_v,$$

where the product is fibered over the common moduli space $\overline{\mathcal{M}}_{0,\Delta_w}$ of target curves.

The fibered product $\overline{\mathcal{H}}_w$ is itself a moduli space of possibly disconnected admissible covers, admitting a map $\overline{\pi}_w^{\text{source}}$ to the moduli space $\prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{M}}_{0,\Delta_v}$ of source curves and a finite flat map $\overline{\pi}_w^{\text{target}}$ to the moduli space $\overline{\mathcal{M}}_{0,\Delta_w}$ of target curves. The stratum G_γ is isomorphic to $\prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{H}}_w$.

Recall that the boundary stratum T_τ in $\overline{\mathcal{M}}_{0,\mathbf{B}}$ is isomorphic to $\prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{M}}_{0,\Delta_w}$. We have

$$\begin{array}{ccc} \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{H}}_w & \xrightarrow{\prod_{w \in \mathcal{V}(\tau)} \overline{\pi}_w^{\text{target}}} & \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{M}}_{0,\Delta_w} \\ \cong \downarrow & & \downarrow \cong \\ G_\gamma & \xrightarrow{\overline{\pi}_{\mathbf{B}}|_{G_\gamma}} & T_\tau \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}} & \xrightarrow{\overline{\pi}_{\mathbf{B}}} & \overline{\mathcal{M}}_{0,\mathbf{B}} \end{array}$$

All rightwards arrows are finite flat maps to moduli spaces of target curves.

Similarly, the stratum S_σ in $\overline{\mathcal{M}}_{0,\mathbf{A}}$ is isomorphic to $\prod_{v \in \mathcal{V}(\sigma)} \overline{\mathcal{M}}_{0,\Delta_v}$, and we have

$$\begin{array}{ccc} \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{H}}_w & \xrightarrow{\prod_{w \in \mathcal{V}(\tau)} \overline{\pi}_w^{\text{source}}} & \prod_{w \in \mathcal{V}(\tau)} \prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{M}}_{0,\Delta_v} \\ \cong \downarrow & & \parallel \\ & & \prod_{v \in \mathcal{V}(\sigma)} \overline{\mathcal{M}}_{0,\Delta_v} \\ G_\gamma & \xrightarrow{\overline{\pi}_{\mathbf{A}}|_{G_\gamma}} & S_\sigma \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}} & \xrightarrow{\overline{\pi}_{\mathbf{A}}} & \overline{\mathcal{M}}_{0,\mathbf{A}} \end{array}$$

All rightwards arrows are maps to moduli spaces of source curves.

G_γ is not necessarily irreducible. Its irreducible components are of the form

$$J = \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{J}}_w,$$

where $\overline{\mathcal{J}}_w$ is an irreducible component of $\overline{\mathcal{H}}_w$.

7.4. Cohomology and Chow groups

Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ be as in the statement of Theorem 7.5. Although $\overline{\mathcal{H}}$ is not smooth, there is a pullback map of Chow groups $(\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow A_k(\overline{\mathcal{H}})$ ([12], Section 6.2). We thus have

$$(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow A_k(\overline{\mathcal{M}}_{0,\mathbf{A}}).$$

For any resolution of singularities $\chi : \tilde{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$, $\chi_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow A_k(\overline{\mathcal{H}})$ agrees with $(\overline{\pi_{\mathbf{B}}})^*$ ([12], Theorem 6.2a). Thus $(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^*$ agrees with $(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^*$ as maps from $A_k(\overline{\mathcal{M}}_{0,\mathbf{B}})$ to $A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$.

By [18], the ‘cycle class’ map gives canonical isomorphisms $A_k(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N}, \mathbb{Z})$. Under these isomorphisms,

$$(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$$

is identified with

$$(\overline{\pi_{\mathbf{A}}} \circ \chi)_* \circ (\overline{\pi_{\mathbf{B}}} \circ \chi)^* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z}).$$

Thus the pushforward $[\mathcal{H}]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Z}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Z})$ may be identified with $(\overline{\pi_{\mathbf{A}}})_* \circ (\overline{\pi_{\mathbf{B}}})^* : A_k(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow A_k(\overline{\mathcal{M}}_{0,\mathbf{A}})$. An analogous statement holds for $[\Gamma]_*$, where Γ is any union of connected components of \mathcal{H} . We use this identification in Proposition 8.1 and Theorem 9.6.

8. Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,N}$

The admissible covers compactification $\overline{\mathcal{H}}$ of $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ naturally lives over the spaces $\overline{\mathcal{M}}_{0,N}$ of stable curves: it extends the “target curve” map to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and the “source curve” map to $\overline{\mathcal{M}}_{0,\mathbf{A}}$. We treat the Hurwitz correspondence \mathcal{H} as a multivalued map from $\mathcal{M}_{0,\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{A}}$. More precisely, \mathcal{H} induces a map from $\mathcal{M}_{0,\mathbf{B}}$ to $\text{Sym}^{(\deg \pi_{\mathbf{B}})} \mathcal{M}_{0,\mathbf{A}}$. Since $\overline{\pi_{\mathbf{B}}} : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{M}}_{0,\mathbf{B}}$ is finite and flat, this extends to a map from $\overline{\mathcal{M}}_{0,\mathbf{B}}$ to $\text{Sym}^{(\deg \pi_{\mathbf{B}})} \overline{\mathcal{M}}_{0,\mathbf{A}}$. Thus the rational correspondence $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : \overline{\mathcal{M}}_{0,\mathbf{B}} \rightrightarrows \overline{\mathcal{M}}_{0,\mathbf{A}}$ may be treated as a regular correspondence. This is supported by the fact that Hurwitz correspondences are homologically composable on the stable curves spaces $\overline{\mathcal{M}}_{0,N}$. The following proposition is the result of a collaboration with Sarah Koch and David Speyer.

Proposition 8.1 (Koch, Ramadas, Speyer). *Let $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,N_1} \rightrightarrows \overline{\mathcal{M}}_{0,N_2}$ and $(\Gamma', \pi'_2, \pi'_3) : \overline{\mathcal{M}}_{0,N_2} \rightrightarrows \overline{\mathcal{M}}_{0,N_3}$ be Hurwitz correspondences. Then for all k ,*

$$[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$$

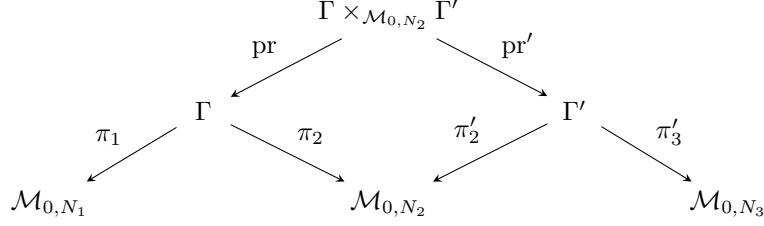
as maps from $H_{2k}(\overline{\mathcal{M}}_{0,N_1})$ to $H_{2k}(\overline{\mathcal{M}}_{0,N_3})$.

Proof. We prove instead that $[\Gamma' \circ \Gamma]_* = [\Gamma']_* \circ [\Gamma]_*$ as maps from $A_k(\overline{\mathcal{M}}_{0,N_1})$ to $A_k(\overline{\mathcal{M}}_{0,N_3})$ (see Section 7.4).

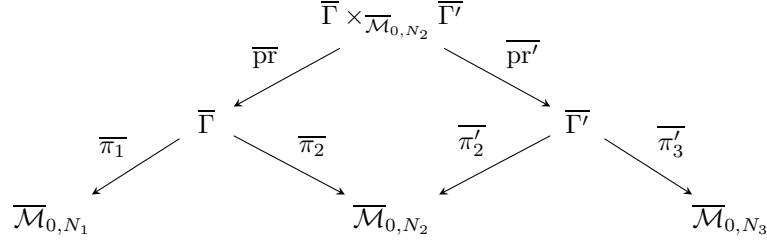
By Lemma 7.2, we may reduce to the case in which Γ and Γ' are unions of connected components of fully marked Hurwitz spaces \mathcal{H} and \mathcal{H}' respectively. The maps π_1 and π'_2 to the moduli spaces of target curves \mathcal{M}_{0,N_1} and \mathcal{M}_{0,N_2} are finite covering maps. Also we have $\pi_2(\Gamma) \subseteq \mathcal{M}_{0,N_2}$. Applying the definition of composition of rational correspondences given in Section 4.1, we may set

$$\Gamma' \circ \Gamma = \Gamma \times_{\mathcal{M}_{0,N_2}} \Gamma',$$

with maps pr and pr' to Γ and Γ' respectively. The map pr is the pullback of a covering map, so is itself a covering map. Let $\overline{\Gamma}$ and $\overline{\Gamma}'$ be the closures of Γ and Γ' in the admissible covers spaces $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}'}$. We have the diagram



We also have the diagram of compactifications:



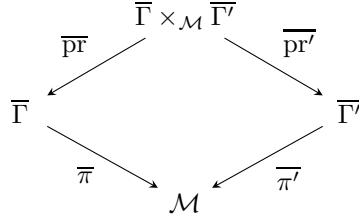
Since $\bar{\pi}_1$ and $\bar{\pi}_2'$ are finite and flat, so is $\bar{\pi}_1 \circ \bar{\text{pr}}$. This means that $\bar{\Gamma} \times_{\bar{\mathcal{M}}_{0,N_2}} \bar{\Gamma}'$ has no irreducible components supported over the boundary $\bar{\mathcal{M}}_{0,N_1} \setminus \mathcal{M}_{0,N_1}$. There is an inclusion $\Gamma \times_{\mathcal{M}_{0,N_2}} \Gamma' \hookrightarrow \bar{\Gamma} \times_{\bar{\mathcal{M}}_{0,N_2}} \bar{\Gamma}'$ whose image is $(\bar{\pi}_1 \circ \bar{\text{pr}})^{-1}(\mathcal{M}_{0,N_1})$. By the above this is an inclusion as a dense open set. We therefore have

$$\begin{aligned}
[\Gamma' \circ \Gamma]_* &= [\bar{\Gamma} \times_{\bar{\mathcal{M}}_{0,N_2}} \bar{\Gamma}']_* \\
&= (\bar{\pi}_3' \circ \bar{\text{pr}}')_* \circ (\bar{\pi}_1 \circ \bar{\text{pr}})^* \\
&= (\bar{\pi}_3')_* \circ (\bar{\text{pr}}')_* \circ (\bar{\text{pr}})^* \circ (\bar{\pi}_1)^*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\Gamma']_* \circ [\Gamma]_* &= [\bar{\Gamma}']_* \circ [\bar{\Gamma}]_* \\
&= (\bar{\pi}_3')_* \circ (\bar{\pi}_2')^* \circ (\bar{\pi}_2)_* \circ (\bar{\pi}_1)^* \\
&= (\bar{\pi}_3')_* \circ (\bar{\text{pr}}')_* \circ (\bar{\text{pr}})^* \circ (\bar{\pi}_1)^* \\
&= [\Gamma' \circ \Gamma]_*.
\end{aligned}$$

Here, the third equality follows from the fact (Proposition 1.7 in [12]) that for any fibered square of varieties



where $\bar{\pi}'$ is a flat map and $\bar{\pi}$ is proper, we have

$$(\bar{\pi}')^* \circ (\bar{\pi})_* = (\bar{\text{pr}}')^* \circ (\bar{\text{pr}})^*.$$

□

By duality of pushforward and pullback, and the fact (Keel, [18]) that $H^{2k}(\overline{\mathcal{M}}_{0,N}) = H^{k,k}(\overline{\mathcal{M}}_{0,N})$, we obtain:

Corollary 8.2 (Koch, Ramadas, Speyer). *Let $(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,N} \rightrightarrows \overline{\mathcal{M}}_{0,N}$ be a dominant Hurwitz self-correspondence. Then Γ is algebraically stable, and its k th dynamical degree is the absolute value of the dominant eigenvalue of $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N})$*

Remark 8.3. Let $(\Gamma, \pi_1, \pi_2) : \mathcal{M}_{0,N} \rightrightarrows \mathcal{M}_{0,N}$ be as above, and let $\overline{\Gamma}$ be the admissible covers compactification of Γ , with its maps $\overline{\pi}_1$ and $\overline{\pi}_2$ to $\overline{\mathcal{M}}_{0,N}$. The map $\overline{\pi}_1$ is flat, so $(\overline{\pi}_1)^*$ takes effective classes on $\overline{\mathcal{M}}_{0,N}$ to effective classes on $\overline{\Gamma}$. Pushforwards always preserve effectiveness, so $[\Gamma]_* = (\overline{\pi}_2)_* \circ (\overline{\pi}_1)^* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N})$ preserves the cone of effective classes. By continuity, $[\Gamma]_*$ preserves the *pseudoeffective cone*, namely the closure of the cone of effective classes. The pseudoeffective cone of any projective variety is a closed cone with nonempty interior that contains no lines ([2, 11]). It follows from the theory of cone-preserving operators ([29]) that $[\Gamma]_*$ has a nonnegative real dominant eigenvalue, with a pseudoeffective eigenvector.

9. Hurwitz correspondences preserve a natural filtration of $H_{2k}(\overline{\mathcal{M}}_{0,N})$

Since Hurwitz correspondences are algebraically stable on $\overline{\mathcal{M}}_{0,N}$, their dynamical degrees are dominant eigenvalues of pushforward maps induced on the homology groups of $\overline{\mathcal{M}}_{0,N}$ (Corollary 8.2). Here, we describe these pushforward maps: we show that they preserve a natural combinatorially defined filtration. We recall the notation and terminology introduced in Section 6.1.

Definition 9.1. Let S_σ be a k -dimensional boundary stratum in $\overline{\mathcal{M}}_{0,N}$. Then $\sum_{v \in \mathcal{V}(\sigma)} \text{md}(v) = k$. Set λ_σ to be the multiset

$$(\text{md}(v))_{v \in \mathcal{V}(\sigma) \text{ with } \text{md}(v) \neq 0}.$$

Then λ_σ is a partition of k ; we say λ_σ is the partition **induced by** the stratum S_σ .

S_σ is isomorphic to $\prod_{v \in \mathcal{V}(\sigma)} \overline{\mathcal{M}}_{0,\Delta_v}$. Each factor $\overline{\mathcal{M}}_{0,\Delta_v}$ has dimension $\text{md}(v)$, so λ_σ encodes the nonzero dimensions of the factors when we write S_σ as this product.

Definition 9.2. For a partition λ of k let $\Lambda_N^{\leq \lambda}$ be the subspace of $H_{2k}(\overline{\mathcal{M}}_{0,N})$ generated by k -dimensional boundary strata that induce the partition λ or any refinement. For \mathbf{P} a finite set we define $\Lambda_{\mathbf{P}}^{\leq \lambda} \subseteq H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ analogously.

If λ_1 is a refinement of λ_2 , we write $\lambda_1 \leq \lambda_2$. This is a partial ordering on the set of partitions of k . Clearly if $\lambda_1 \leq \lambda_2$, then $\Lambda_N^{\leq \lambda_1} \subseteq \Lambda_N^{\leq \lambda_2}$.

The dual tree of a k -dimensional boundary stratum S_σ in $\overline{\mathcal{M}}_{0,N}$ has $N - k - 2$ vertices. This means that the partition λ_σ has at most $N - k - 2$ parts. Conversely, any partition of k with at most $N - k - 2$ parts arises from a boundary stratum in $\overline{\mathcal{M}}_{0,N}$. We say that λ is **realizable** on $\mathcal{M}_{0,N}$ if λ has at most $N - k - 2$ parts.

Lemma 9.3. *Let S_{σ_0} be a k -dimensional boundary stratum inducing partition λ_{σ_0} of k . Then there does not exist an equality in $H_{2k}(\overline{\mathcal{M}}_{0,N})$ of the form*

$$[S_{\sigma_0}] = \sum_j \beta_j [S_{\sigma(j)}],$$

where every partition $\lambda_{\sigma(j)}$ is distinct from λ_{σ_0} .

Proof. We recall the additive relations $R(\sigma, v, i_1, \dots, i_4)$ introduced in Section 6.1 (Equation 2). We can rewrite

$$R(\sigma, v, i_1, \dots, i_4) = \sum_{(\Delta_1, \Delta_2)} ([S(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)] - [S(i_1 i_3 \Delta_1 | \Delta_2 i_2 i_4)]).$$

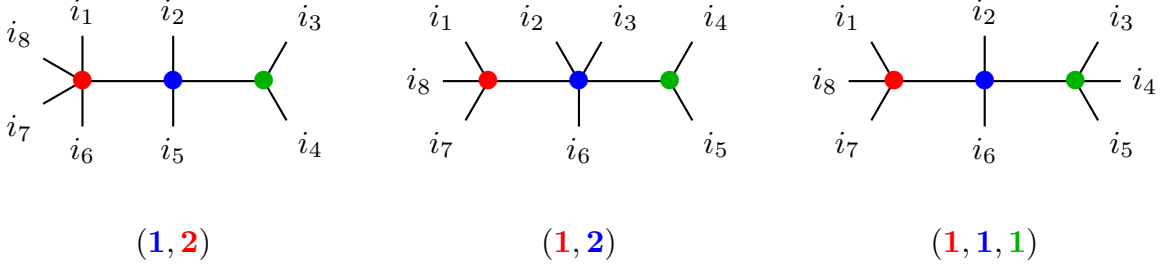


Figure 2: Dual trees of boundary strata generating the subspace $\Lambda_8^{\leq(1,2)} \subseteq H_6(\overline{\mathcal{M}}_{0,8})$

It is clear from definitions that $S(i_1 i_2 \Delta_1 | \Delta_2 i_3 i_4)$ and $S(i_1 i_3 \Delta_1 | \Delta_2 i_2 i_4)$ induce the same partition of k , and they appear in $R(\sigma, v, i_1, \dots, i_4)$ paired up and with opposite signs. Therefore for any partition λ of k , the coefficients in $R(\sigma, v, i_1, \dots, i_4)$ of boundary strata inducing λ sum to zero. Since the relations $R(\sigma, v, i_1, \dots, i_4)$ generate all relations among boundary strata, we conclude that for any additive relation R , the coefficients of boundary strata inducing a fixed partition λ sum to zero. Therefore there is no equality $[S_{\sigma_0}] = \sum_j \beta_j [S_{\sigma(j)}]$, where each partition $\lambda_{\sigma(j)}$ is different from λ_{σ_0} . \square

We deduce:

Corollary 9.4. (i) For S_σ a k -dimensional boundary stratum and λ a partition of k , $S_\sigma \in \Lambda_N^{\leq \lambda}$ if and only if $\lambda_\sigma \leq \lambda$.

(ii) For λ_1 and λ_2 distinct realizable partitions of k , $\Lambda_N^{\leq \lambda_1} \neq \Lambda_N^{\leq \lambda_2}$.

The collection of subspaces $\{\Lambda_N^{\leq \lambda}\}_\lambda$ is a poset-filtration for $H_{2k}(\overline{\mathcal{M}}_{0,N})$ indexed by $\{\text{partitions of } k\}$ with the refinement partial ordering. Denote by (k) the one-part partition of k . Then $\Lambda_N^{\leq (k)} = H_{2k}(\overline{\mathcal{M}}_{0,N})$.

Forgetful maps respect the filtration $\{\Lambda_N^{\leq \lambda}\}_\lambda$:

Lemma 9.5. Let $N \geq N' \geq 3$. Then, if $\mu : \overline{\mathcal{M}}_{0,N} \rightarrow \overline{\mathcal{M}}_{0,N'}$ is the forgetful map sending (C, p_1, \dots, p_N) to $(C, p_1, \dots, p_{N'})$, the pushforward $\mu_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N'})$ sends $\Lambda_N^{\leq \lambda}$ to $\Lambda_{N'}^{\leq \lambda}$ for all partitions λ of k .

Proof. We may assume $N' = N - 1$. Let S_σ be any k -dimensional boundary stratum in $\overline{\mathcal{M}}_{0,N}$, inducing partition λ_σ of k . We show $\mu_*([S_\sigma]) \in \Lambda_{N'}^{\leq \lambda_\sigma}$. The image $\mu(S_\sigma) = S_{\sigma'}$ is a boundary stratum in $\overline{\mathcal{M}}_{0,N'}$, where σ' is obtained from σ by deleting the leg ℓ_N and stabilizing as in Section 6.1. Let v_0 on σ be the vertex with the leg ℓ_N .

Case 1. $|v_0| \geq 4$.

The N' -marked tree obtained by deleting ℓ_N is stable, and therefore is σ' . The moduli dimension of v_0 is positive, and the corresponding vertex on σ' has moduli dimension one less. All other vertices of σ' have the same moduli dimension as the corresponding vertices of σ , so $\dim S_{\sigma'} = \dim S_\sigma - 1$. Therefore $\mu_*([S_\sigma]) = 0 \in \Lambda_{N'}^{\leq \lambda_\sigma}$.

Case 2. $|v_0| = 3$ and Δ_{v_0} consists of ℓ_N , an edge e_1 connecting v_0 to v_1 , and an edge e_2 connecting v_0 to v_2 .

The stabilization σ' is obtained from σ by deleting ℓ_N , v_0 , e_1 , and e_2 , and connecting v_1 and v_2 with a new edge. The tree σ' does not have the vertex v_0 , which has moduli dimension zero, however all other vertices on σ' have the same moduli dimension as the corresponding vertices on σ . So $\dim S_{\sigma'} = \dim S_\sigma$ and $\lambda_{\sigma'} = \lambda_\sigma$. Therefore $\mu_*([S_\sigma]) = [S_{\sigma'}] \in \Lambda_{N'}^{\leq \lambda_\sigma}$.

Case 3. $|v_0| = 3$ and Δ_{v_0} consists of ℓ_N , another leg ℓ_i , and an edge e connecting v_0 to v_1 .

The stabilization σ' is obtained from σ by deleting ℓ_N , v_0 , and e , and attaching the leg ℓ_i to v_1 . The tree σ' does not have the vertex v_0 , which has moduli dimension zero, however all other vertices on σ' have the same moduli dimension as the corresponding vertices on σ . So $\dim S_{\sigma'} = \dim S_\sigma$ and $\lambda_{\sigma'} = \lambda_\sigma$. Therefore $\mu_*([S_\sigma]) = [S_{\sigma'}] \in \Lambda_{N'}^{\leq \lambda_\sigma}$. \square

Our main result in this section is Theorem 9.6, showing that the filtration $\{\Lambda_N^{\leq \lambda}\}_\lambda$ is preserved by all Hurwitz correspondences. Our proof uses the stratification of the moduli spaces of admissible covers (Section 7.3).

Theorem 9.6. *Let $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : \overline{\mathcal{M}}_{0,\mathbf{B}} \dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{A}}$ be a Hurwitz correspondence. Then for every partition λ of k , $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}})$ sends $\Lambda_{\mathbf{B}}^{\leq \lambda}$ to $\Lambda_{\mathbf{A}}^{\leq \lambda}$.*

Proof of Theorem 9.6. By Lemma 7.2, we may assume that Γ is a union of connected components of a fully marked Hurwitz space \mathcal{H} , and by Lemma 9.5, we may assume that $\mathbf{A} = \mathbf{A}^{\text{full}}$ as in Section 7.1.

Let $\overline{\mathcal{H}}$ be the admissible covers compactification of \mathcal{H} and let $\overline{\Gamma}$ be the closure of Γ in $\overline{\mathcal{H}}$. The compactifications $\overline{\mathcal{H}}$ and $\overline{\Gamma}$ both have maps $\overline{\pi}_{\mathbf{B}}$ and $\overline{\pi}_{\mathbf{A}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$. As in Section 7.4, we have $[\Gamma]_* = (\overline{\pi}_{\mathbf{A}})_* \circ (\overline{\pi}_{\mathbf{B}})^*$.

Fix T_τ any k -dimensional boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{B}}$. Let λ_τ be the partition of k induced by T_τ . We show $[\Gamma]_*([T_\tau]) \in \Lambda_{\mathbf{A}}^{\leq \lambda_\tau}$. Since $\overline{\pi}_{\mathbf{B}}$ is flat,

$$(\overline{\pi}_{\mathbf{B}})^*([T_\tau]) = \sum_J m_J [J],$$

where the sum is over irreducible components J of the preimage $(\overline{\pi}_{\mathbf{B}})^{-1}(T_\tau)$ in $\overline{\Gamma}$, and m_J is a positive integer multiplicity.

To show that $(\overline{\pi}_{\mathbf{A}})_* \circ (\overline{\pi}_{\mathbf{B}})^*([T_\tau])$ is in $\Lambda_{\mathbf{A}}^{\leq \lambda_\tau}$, it suffices to show that for J an irreducible component of $(\overline{\pi}_{\mathbf{B}})^{-1}(T_\tau)$, the pushforward $(\overline{\pi}_{\mathbf{A}})_*([J])$ is in $\Lambda_{\mathbf{A}}^{\leq \lambda_\tau}$.

Fix such a J — it is an irreducible component of a boundary stratum G_γ of $\overline{\mathcal{H}}$, with combinatorial type $\gamma = (\sigma, \tau, d^{\text{vert}}, f^{\text{vert}}, (F_v)_{v \in \mathcal{V}(\sigma)}, (\text{br}_v)_{v \in \mathcal{V}(\sigma)}, (\text{rm}_v)_{v \in \mathcal{V}(\sigma)})$ (Section 7.3). There is a decomposition $G_\gamma = \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{H}}_w$, where $\overline{\mathcal{H}}_w$ is an admissible covers space of dimension $\text{md}(w)$, and a decomposition

$$J = \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{J}}_w,$$

where $\overline{\mathcal{J}}_w$ is an irreducible component of $\overline{\mathcal{H}}_w$. Each factor $\overline{\mathcal{J}}_w$ admits a map $\overline{\pi}_w^{\text{source}}$ to $\prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{M}}_{0,\Delta_v}$ and a finite flat map $\overline{\pi}_w^{\text{target}}$ to $\overline{\mathcal{M}}_{0,\Delta_w}$. We conclude that $\dim \overline{\mathcal{J}}_w = \dim \overline{\mathcal{M}}_{0,\Delta_w} = \text{md}(w)$. The key observation of our proof is that this decomposition of J also induces the partition λ_τ .

The cohomology (respectively Chow) ring of $\prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{M}}_{0,\Delta_v}$ is the tensor product of the cohomology (respectively Chow) rings of the factors, which in turn are generated by the classes of boundary strata. Thus we may write

$$(\overline{\pi}_w^{\text{source}})_*([\overline{\mathcal{J}}_w]) = \sum_{q \in Q_w} \beta_q \bigotimes_{v \in (f^{\text{vert}})^{-1}(w)} [S_{wv}^q],$$

where Q_w is a finite set, β_q is an integer multiplicity, and S_{wv}^q is a boundary stratum of dimension k_{wv}^q in $\overline{\mathcal{M}}_{0,\Delta_v}$. Denote by λ_{wv}^q the partition of k_{wv}^q induced by S_{wv}^q . For every $q \in Q_w$,

$$\sum_{v \in (f^{\text{vert}})^{-1}(w)} k_{wv}^q = \dim \overline{\mathcal{J}}_w = \text{md}(w),$$

so $\bigcup_{v \in (f^{\text{vert}})^{-1}(w)} \lambda_{wv}^q$ is a partition of $\text{md}(w)$. The map $\overline{\pi}_{\mathbf{A}}|_J$ decomposes as a product:

$$J = \prod_{w \in \mathcal{V}(\tau)} \overline{\mathcal{J}}_w \xrightarrow{\prod_{w \in \mathcal{V}(\tau)} \overline{\pi}_w^{\text{source}}} \prod_{w \in \mathcal{V}(\tau)} \prod_{v \in (f^{\text{vert}})^{-1}(w)} \overline{\mathcal{M}}_{0,\Delta_v} \cong S_\sigma \xrightarrow{\iota} \overline{\mathcal{M}}_{0,\mathbf{A}},$$

so

$$\begin{aligned}
(\overline{\pi_{\mathbf{A}}|_J})_*([J]) &= \iota_* \left(\bigotimes_{w \in \mathcal{V}(\tau)} (\overline{\pi_w^{\text{source}}})_*([\mathcal{J}_w]) \right) \\
&= \iota_* \left(\bigotimes_{w \in \mathcal{V}(\tau)} \left(\sum_{q \in Q_w} \beta_q \bigotimes_{v \in (f^{\text{vert}})^{-1}(w)} [S_{wv}^q] \right) \right) \\
&= \iota_* \left(\sum_{(q(w))_w \in \prod_{w \in \mathcal{V}(\tau)} Q_w} \left(\prod_{w \in \mathcal{V}(\tau)} \beta_{q(w)} \right) \bigotimes_{w \in \mathcal{V}(\tau)} \bigotimes_{v \in (f^{\text{vert}})^{-1}(w)} [S_{wv}^{q(w)}] \right) \\
&= \sum_{(q(w))_w \in \prod_{w \in \mathcal{V}(\tau)} Q_w} \left(\prod_{w \in \mathcal{V}(\tau)} \beta_{q(w)} \right) \left[\iota \left(\prod_{w \in \mathcal{V}(\tau)} \prod_{v \in (f^{\text{vert}})^{-1}(w)} S_{wv}^{q(w)} \right) \right].
\end{aligned}$$

For fixed $(q(w))_w$, the image

$$\iota \left(\prod_{w \in \mathcal{V}(\tau)} \prod_{v \in (f^{\text{vert}})^{-1}(w)} S_{wv}^{q(w)} \right)$$

under the gluing morphism is a k -dimensional boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{A}}$. It induces partition of k :

$$\bigcup_{w \in \mathcal{V}(\tau)} \bigcup_{v \in (f^{\text{vert}})^{-1}(w)} \lambda_{wv}^{q(w)}.$$

As expressed this is clearly a refinement of

$$\lambda_\tau = (\text{md}(w))_w \in \mathcal{V}(\tau) \text{ with } \text{md}(w) \neq 0.$$

Thus $(\overline{\pi_{\mathbf{A}}})_*([J])$ is in $\Lambda_{\mathbf{A}}^{\leq \lambda_\tau}$. □

10. Dynamical degrees of Hurwitz correspondences

Theorem 9.6 implies that $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}})$ can be written, in multiple different ways, as a block lower triangular matrix. In the case of a Hurwitz self-correspondence, one can ask which of these blocks contains the dynamical degree — the dominant eigenvalue. This section addresses this question.

Set

$$\Lambda_N^{<(k)} := \sum_{\lambda \text{ has } \geq 2 \text{ parts}} \Lambda_N^{\leq \lambda}$$

and

$$\Omega_N^k := \frac{H_{2k}(\overline{\mathcal{M}}_{0,N})}{\Lambda_N^{<(k)}}.$$

We prove:

Theorem 10.6. *Let $\Gamma : \overline{\mathcal{M}}_{0,N} \dashrightarrow \overline{\mathcal{M}}_{0,N}$ be a dominant Hurwitz correspondence. Then the k th dynamical degree of Γ is the absolute value of the dominant eigenvalue of $[\Gamma]_* : \Omega_N^k \rightarrow \Omega_N^k$.*

10.1. Hurwitz correspondences on alternate compactifications of $\mathcal{M}_{0,N}$

Definition 10.1. A weight datum $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ is called **minimal** if, for every subset $P \subseteq \{1, \dots, N\}$, $\sum_{i \in P} \epsilon_i > 1$ if and only if $\sum_{i \notin P} \epsilon_i < 1$.

Lemma 10.2. *Let $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ be a minimal weight datum. Then every N -marked stable tree σ has a unique vertex that is ϵ -stable.*

Proof. Every tree has at least one ϵ -stable vertex (Section 6.2). Conversely, let σ be an N -marked stable tree. If σ has a unique vertex, we are done. If not, any edge e disconnects σ into two components σ_1 and σ_2 . Since ϵ is minimal, exactly one of $\sum_{\ell_i \text{ on } \sigma_1} \epsilon_i$ and $\sum_{\ell_i \text{ on } \sigma_2} \epsilon_i$ is greater than 1. If $\sum_{\ell_i \text{ on } \sigma_1} \epsilon_i > 1$, then no vertex on σ_2 is ϵ -stable, and if $\sum_{\ell_i \text{ on } \sigma_2} \epsilon_i > 1$, then no vertex on σ_1 is ϵ -stable. Since e was arbitrary, σ has at most one ϵ -stable vertex. \square

Proposition 10.3. *Let ϵ be a minimal weight datum, and let $\rho_\epsilon : \overline{\mathcal{M}}_{0,N} \rightarrow \overline{\mathcal{M}}_{0,N}(\epsilon)$ be the reduction morphism. Then $\ker((\rho_\epsilon)_*) \subseteq H_{2k}(\overline{\mathcal{M}}_{0,N})$ contains $\Lambda_N^{<(k)}$.*

Proof. Let S_σ be any boundary stratum in $\Lambda_N^{<(k)}$. The partition λ_σ has at least two parts, so σ has at least two vertices with positive moduli dimension. By Lemma 10.2, at least one of these is not ϵ -stable, and so by Lemma 6.12, $S_\sigma \in \ker((\rho_\epsilon)_*)$. \square

There are many minimal weight data, giving rise to non-isomorphic spaces of weighted stable curves, as follows.

Example 10.4. $\epsilon_1 = 1 + \frac{N}{10^N}$, and $\epsilon_i = \frac{1}{N-1} - \frac{1}{10^N}$ for $i = 2, \dots, N$. Then $\overline{\mathcal{M}}_{0,N}(\epsilon) \cong \mathbb{P}^{N-3}$

Example 10.5. Let ϵ^\dagger be the following minimal weight datum:

- N **odd.** $\epsilon_i = \frac{2}{N} + \frac{1}{10^N}$ for all i .
- N **even.** $\epsilon_1 = \frac{2}{N} + \frac{1}{10^N}$, and $\epsilon_2 = \dots = \epsilon_N = \frac{2}{N} - \frac{1}{N \cdot 10^N}$.

The subsets of $\{\epsilon_1, \dots, \epsilon_N\}$ with sum greater than 1 are those with size more than $N/2$, or those with size $N/2$ containing ϵ_1 . Set $X_N^\dagger = \overline{\mathcal{M}}_{0,N}(\epsilon^\dagger)$ and $\rho^\dagger : \overline{\mathcal{M}}_{0,N} \rightarrow X_N^\dagger$ the reduction morphism. X_N^\dagger has Picard rank N ; in particular, it is not isomorphic to \mathbb{P}^{N-3} .

There is a simple description of an ϵ^\dagger -stable curve: C an irreducible genus zero curve, and p_1, \dots, p_N marked points, not necessarily distinct, such that

- at least 3 distinct points on C are marked,
- no point on C has greater than $N/2$ marks, and
- if a point on C has exactly $N/2$ marks, then p_1 is not among them.

By Proposition 10.3 and Lemma 4.8, we obtain:

Theorem 10.6. *Let $\Gamma : \overline{\mathcal{M}}_{0,N} \dashrightarrow \overline{\mathcal{M}}_{0,N}$ be a dominant Hurwitz correspondence. Then the k th dynamical degree of Γ is the absolute value of the dominant eigenvalue of $[\Gamma]_* : \Omega_N^k \rightarrow \Omega_N^k$.*

Thus the dynamical degrees of Γ may be computed without any information about its action on the subspace $\Lambda_N^{<(k)}$. This suggests that this action is irrelevant to the dynamics of Γ on $\mathcal{M}_{0,N}$. Relatedly, we have

Proposition 10.7. *Any effective cycle $\alpha \in \Lambda_N^{<(k)}$ is supported on the boundary of $\overline{\mathcal{M}}_{0,N}$.*

Proof. By Proposition 10.3, α is in the kernel of $\rho_*^\dagger : \overline{\mathcal{M}}_{0,N} \rightarrow X_N^\dagger$, where X_N^\dagger is as in Example 10.5. The interior $\mathcal{M}_{0,N}$ maps isomorphically onto an open set on X_N^\dagger . Thus the support of α does not intersect $\mathcal{M}_{0,N}$. \square

Corollary 10.8. *Let $\Gamma : \overline{\mathcal{M}}_{0,N} \dashrightarrow \overline{\mathcal{M}}_{0,N}$ be a Hurwitz correspondence and $S_\sigma \in \Lambda_N^{<(k)}$ be a boundary stratum. Then the cycle $[\Gamma]_*([S_\sigma])$ is supported on the boundary of $\overline{\mathcal{M}}_{0,N}$.*

Interestingly, for $k > \frac{\dim \mathcal{M}_{0,N}}{2}$, we can contract these strata and keep k -stability of Γ (Corollary 10.11).

Lemma 10.9. *Let $\overline{\mathcal{M}}_{0,N}(\epsilon)$ be a moduli space of weighted stable curves, with reduction morphism $\rho_\epsilon : \overline{\mathcal{M}}_{0,N} \rightarrow \overline{\mathcal{M}}_{0,N}(\epsilon)$. Then the kernel of $(\rho_\epsilon)_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N}(\epsilon))$ is generated by classes of k -dimensional boundary strata.*

Proof. The homology groups of $\overline{\mathcal{M}}_{0,N}$ and $\overline{\mathcal{M}}_{0,N}(\epsilon)$ are generated by the classes of boundary strata. Denote by \mathcal{F} the free \mathbb{R} -vector space on the set of k -dimensional boundary strata in $\overline{\mathcal{M}}_{0,N}$, and denote by \mathcal{F}^ϵ the free \mathbb{R} -vector space on the set of k -dimensional boundary strata in $\overline{\mathcal{M}}_{0,N}(\epsilon)$. Let $\mathcal{R} \subseteq \mathcal{F}$ and $\mathcal{R}^\epsilon \subseteq \mathcal{F}^\epsilon$ be the subspaces of relations, i.e. linear combinations of boundary strata that are homologous to zero in $\overline{\mathcal{M}}_{0,N}$ and $\overline{\mathcal{M}}_{0,N}(\epsilon)$ respectively. Define a map $\tilde{\rho}_\epsilon : \mathcal{F} \rightarrow \mathcal{F}^\epsilon$ on generators as follows. If $\dim(\rho_\epsilon(S_\sigma)) < k$, set $\tilde{\rho}_\epsilon(S_\sigma) = 0$, else set $\tilde{\rho}_\epsilon(S_\sigma) = \rho_\epsilon(S_\sigma)$. We have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{F} & \longrightarrow & H_{2k}(\overline{\mathcal{M}}_{0,N}) \longrightarrow 0 \\ & & \tilde{\rho}_\epsilon \downarrow & & \tilde{\rho}_\epsilon \downarrow & & (\rho_\epsilon)_* \downarrow \\ 0 & \longrightarrow & \mathcal{R}^\epsilon & \longrightarrow & \mathcal{F}^\epsilon & \longrightarrow & H_{2k}(\overline{\mathcal{M}}_{0,N}(\epsilon)) \longrightarrow 0 \end{array}$$

Ceyhan [5] gives generators for \mathcal{R}^ϵ . Each generator is the image under $\tilde{\rho}_\epsilon$ of some generator $\mathcal{R}(\sigma, v, i_1, \dots, i_4)$ in \mathcal{R} (see Section 6.1). Thus $\tilde{\rho}_\epsilon(\mathcal{R}) = \mathcal{R}^\epsilon$, and the kernel of $\mathcal{F} \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N}(\epsilon))$ is the sum $\mathcal{R} + \ker(\tilde{\rho}_\epsilon)$.

On the other hand, any two k -dimensional boundary strata in $\overline{\mathcal{M}}_{0,N}$ with the same k -dimensional image in $\overline{\mathcal{M}}_{0,N}(\epsilon)$ are homologous. Thus $\ker(\tilde{\rho}_\epsilon)/(\ker(\tilde{\rho}_\epsilon) \cap \mathcal{R})$ is generated by boundary strata, so $\ker((\rho_\epsilon)_*)$ is also generated by boundary strata. \square

Proposition 10.10. *Let ϵ^\dagger , X_N^\dagger , and ρ^\dagger be as in Example 10.5. Then for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$,*

$$\Lambda_N^{<(k)} = \ker(\rho_*^\dagger) \subseteq H_{2k}(\overline{\mathcal{M}}_{0,N}).$$

Proof. Fix $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$. By Proposition 10.3, we have $\Lambda_N^{<(k)} \subseteq \ker(\rho_*^\dagger)$. Suppose S_σ is a k -dimensional boundary stratum not in $\Lambda_N^{<(k)}$. Then $\lambda_\sigma = (k)$ and σ has a unique vertex v with positive moduli dimension $\text{md}(v) = k$. So v has valence

$$k + 3 \geq \frac{\dim \mathcal{M}_{0,N}}{2} + 3 = \frac{N + 3}{2} > \frac{N}{2} + 1.$$

Thus, for every flag $\delta \in \Delta_v$, the set $\{i | \delta = \delta(v \rightarrow i)\}$ has size less than $N/2$, so $\sum_{i | \delta = \delta(v \rightarrow i)} \epsilon_i < 1$. We conclude that

$$\sum_{\delta \in \Delta_v} \min \left\{ 1, \sum_{i | \delta = \delta(v \rightarrow i)} \epsilon_i \right\} = \sum_{i=1}^N \epsilon_i > 2,$$

so v is ϵ -stable. By Lemma 6.12, $S_\sigma \notin \ker(\rho_*^\dagger)$.

Thus a k -dimensional boundary stratum is in $\ker(\rho_*^\dagger)$ exactly if it is in $\Lambda_N^{<(k)}$. By Lemma 10.9, $\ker(\rho_*^\dagger) = \Lambda_N^{<(k)}$. \square

By Proposition 10.10 and Lemma 4.9:

Corollary 10.11. *Let $\Gamma : X_N^\dagger \dashrightarrow X_N^\dagger$ be a dominant Hurwitz correspondence. Then for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$:*

(i) *We have a commutative diagram*

$$\begin{array}{ccc}
\Omega_N^k & \xrightarrow{[\Gamma]_*} & \Omega_N^k \\
(\rho^\dagger)_* \cong \downarrow & & \cong \downarrow (\rho^\dagger)_* \\
H_{2k}(X_N^\dagger) & \xrightarrow{[\Gamma]_*} & H_{2k}(X_N^\dagger)
\end{array}$$

- (ii) $[\Gamma]_* : H_{2k}(X_N^\dagger) \rightarrow H_{2k}(X_N^\dagger)$ preserves the cone of effective classes, and
- (iii) Γ is k -stable on X_N^\dagger .

By Theorem 10.6, the dynamical degree of a Hurwitz correspondence $\Gamma : \overline{\mathcal{M}}_{0,N} \dashrightarrow \overline{\mathcal{M}}_{0,N}$ is the absolute value of the dominant eigenvalue of the induced action of Γ on the quotient vector space Ω_N^k . By Corollary 10.11, for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$, this action on Ω_N^k also has an interpretation as the k -stable action of Γ on $H_{2k}(X_N^\dagger)$. We obtain as a corollary:

Corollary 10.12. *For $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$, $[\Gamma]_* : \Omega_N^k \rightarrow \Omega_N^k$ has a nonnegative dominant eigenvalue.*

Proof. By Corollary 10.11 (ii), $[\Gamma]_* : H_{2k}(X_N^\dagger) \rightarrow H_{2k}(X_N^\dagger)$ preserves the cone of effective classes, thus as in Remark 8.3 has a nonnegative dominant eigenvalue. \square

For $k < \frac{\dim \mathcal{M}_{0,N}}{2}$, we ask

- (1) Can we interpret $[\Gamma]_* : \Omega_N^k \rightarrow \Omega_N^k$ as the k -stable homological action of Γ on an alternate smooth compactification of $\mathcal{M}_{0,N}$?
- (2) Equivalently, is there an alternate smooth compactification X_N of $\mathcal{M}_{0,N}$ admitting a birational morphism $\rho : \overline{\mathcal{M}}_{0,N} \rightarrow X_N$ such that the kernel of $\rho_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(X_N)$ is $\Lambda_N^{<(k)}$?
- (3) Is there an alternate smooth compactification X_N of $\mathcal{M}_{0,N}$ such that every Hurwitz correspondence $\Gamma : X_N \dashrightarrow X_N$ is k -stable?

Smyth ([30]) defined a *modular compactification* of $\mathcal{M}_{0,N}$ to be one that extends its moduli space interpretation. (These include $\overline{\mathcal{M}}_{0,N}$ and any space of weighted stable curves.) If the discussion is restricted to modular compactifications, the answers to questions 1 and 2 are both “no.” In fact:

Proposition 10.13 ([24]). *Fix N, k with $1 \leq k < \frac{\dim \mathcal{M}_{0,N}}{2}$, and a nonempty subset $L \subseteq \{\text{partitions of } k\}$. Then there is no smooth or projective modular compactification X_N of $\mathcal{M}_{0,N}$ with reduction morphism $\rho : \overline{\mathcal{M}}_{0,N} \rightarrow X_N$ such that the kernel of $(\rho)_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(X_N)$ is equal to $\sum_{\lambda \in L} \Lambda_N^{\leq \lambda}$.*

There exist other natural classes of compactifications of $\mathcal{M}_{0,N}$ (e.g. [13], [4]). However, any compactification arising in the construction of [13] or [4] that is *not* modular in the sense of Smyth is singular, thus may not be used to study the dynamics of Hurwitz correspondences. This leads us to conjecture that the answers to all three questions are in the negative.

10.2. The filtration $\{\Lambda_N^{\leq \lambda}\}$ and quotients Ω_N^k

In light of Theorems 9.6 and 10.6, we ask: What are the dimensions of the subspaces $\Lambda_N^{\leq \lambda}$ and quotients Ω_N^k ? In particular, the k th dynamical degree of a Hurwitz correspondence Γ , namely the dominant eigenvalue of $[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,N}) \rightarrow H_{2k}(\overline{\mathcal{M}}_{0,N})$, is in fact the dominant eigenvalue of the smaller matrix $[\Gamma]_* : \Omega_N^k \rightarrow \Omega_N^k$. What are the comparative sizes of the two matrices — how much easier does Theorem 10.6 make computation of dynamical degrees?

There is a recursive formula ([18]) for the Betti numbers of $\overline{\mathcal{M}}_{0,N}$. Moon [23, Corollary 5.3.2] gives a formula for the Poincaré polynomial of X_N^\dagger . For $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$, the quotient Ω_N^k is isomorphic to $H_{2k}(X_N^\dagger)$,

so this tells us $\dim \Omega_N^k$ for k in this range. This allows us, in principle, to compare $H_{2k}(\overline{\mathcal{M}}_{0,N})$ and Ω_N^k for $k \geq \frac{\dim \mathcal{M}_{0,N}}{2}$.

For $k = \dim \mathcal{M}_{0,N} - 1$ (divisors) and $k = 1$ (curve classes), we can be more explicit:

The case $k = \dim \mathcal{M}_{0,N} - 1$. The homology group $H_{2k}(\overline{\mathcal{M}}_{0,N}) = \text{Pic}(\overline{\mathcal{M}}_{0,N})$ has dimension $\frac{2^N - N^2 + N - 2}{2}$ and is generated by the classes of boundary divisors. $\overline{\mathcal{M}}_{0,N}$ carries N tautological line bundles:

Definition 10.14. For $i \in \{1, \dots, N\}$, define a line bundle \mathcal{L}_i on $\overline{\mathcal{M}}_{0,N}$ as follows: the fiber over $[C, p_1, \dots, p_N]$ is $T_{p_i}^\vee C$, that is, the cotangent line to C at the marked point p_i .

By [9], we have

- (i) The boundary divisors $\{S_\sigma | \lambda_\sigma \text{ has exactly two parts}\}$ are a basis for $\Lambda_N^{<(k)}$. Thus

$$\dim \Lambda_N^{<(k)} = \frac{1}{2} \sum_{j=3}^{N-3} \binom{N}{j} = \frac{1}{2} (2^N - 2 - 2N - N(N-1)).$$

- (ii) The divisor classes $\{c_1(\mathcal{L}_i)\}_{i=1, \dots, N}$ are a basis for Ω_N^k . Thus $\dim \Omega_N^k = N$.

It follows from Theorem 10.6 that the $(N-4)$ th dynamical degree of Γ is an algebraic integer of degree at most N . This generalizes a result in [20].

The case $k = 1$. By Poincaré duality,

$$\dim H_2(\overline{\mathcal{M}}_{0,N}) = \dim H_{2(\dim \mathcal{M}_{0,N}-1)}(\overline{\mathcal{M}}_{0,N}) = \frac{2^N - N^2 + N - 2}{2}.$$

Since there is only one partition of 1, the filtration $\{\Lambda_N^{<\lambda}\}$ of $H_2(\overline{\mathcal{M}}_{0,N})$ is trivial, and $H_2(\overline{\mathcal{M}}_{0,N}) = \Omega_N^1$.

By Corollary 9.4, for k strictly between 1 and $\dim \mathcal{M}_{0,N}$, $\dim \Omega_N^k < \dim H_{2k}(\overline{\mathcal{M}}_{0,N})$.

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